

# GLOBAL WELL-POSEDNESS OF HELICOIDAL EULER EQUATIONS

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**ABSTRACT.** This paper deals with the global existence and uniqueness results for the three-dimensional incompressible Euler equations with a particular structure for initial data lying in critical spaces. In this case the BKM criterion is not known.

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## 1. INTRODUCTION

The purpose of this paper is to investigate the global well-posedness of the following three-dimensional incompressible Euler system in the whole space with helicoidal initial data. This system is described as follows:

$$(E) \quad \begin{cases} \partial_t u + (u \cdot \nabla)u + \nabla \Pi = 0, & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^3, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u^0. \end{cases}$$

Here, the vector field  $u = (u_1, u_2, u_3)$  is the velocity of the fluid and  $\Pi$  is a scalar pressure function.

The operator  $u \cdot \nabla$  is given explicitly by  $u \cdot \nabla = \sum_{j=1}^3 u_j \partial_j$  and the incompressibility of the fluid is

expressed via the second equation of the system  $\operatorname{div} u = \sum_{j=1}^3 \partial_j u_j = 0$ .

The question of local or global existence and uniqueness of solutions to the system (E) is one of the most important problems in fluid mechanics. Existence and uniqueness theories of (2 or 3 dimensional) Euler equations have been studied by many mathematicians and physicists. W. Wolibner [26] started the subject in Hölder spaces, D. Ebin [11], J. Bourguignon [3], R. Temam [21], T. Kato and G. Ponce [15] worked out this subject in Sobolev spaces. Much of the studies on the Euler equations of an ideal incompressible fluid in Besov spaces has been done by M. Vishik ([23], [24], [25]), D. Chae [6] and C. Park and J. Park [16].

The question of global existence (even for smooth initial data) is still open and continues to be one of the most challenging problems in nonlinear PDEs. The degree of difficulties depends strongly on the dimensions (2 or 3) and the regularity of the initial data. In this context, the vorticity play a fundamental role. In fact, the well-known BKM criterion [4] ensures that the development of finite time singularities for Kato's solutions is related to the blowup of the  $L^\infty$  norm of the vorticity near the maximal time existence. In 2-D, the vorticity satisfies a transport equation

$$\partial_t \omega + (u \cdot \nabla) \omega = 0.$$

In space dimension three, the vorticity satisfies the equation

$$(1) \quad \partial_t \omega + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u$$

and the main difficulty for establishing global regularity is to understand how the vortex-stretching term  $(\omega \cdot \nabla) u$  affects the dynamic of the fluid. While the question of global existence for 3-D Euler system is widely open, some positive results are available for the 3-D flows with some geometry

constraints as the so-called axisymmetric flows without swirl. We say that a vector field  $u$  is axisymmetric if it has the form :

$$u(x, t) = u_r(r, z, t)e_r + u_z(r, z, t)e_z, \quad x = (x_1, x_2, z), \quad r = (x_1^2 + x_2^2)^{\frac{1}{2}},$$

where  $(e_r, e_\theta, e_z)$  is the cylindrical basis of  $\mathbb{R}^3$  and the components  $u_r$  and  $u_z$  do not depend on the angular variable. The main feature of axisymmetric flows arises in the vorticity which takes the form

$$\omega = (\partial_z u^r - \partial_r u^z)e_\theta,$$

and satisfies

$$(2) \quad \partial_t \omega + (u \cdot \nabla) \omega = \frac{u^r}{r} \omega.$$

Consequently the quantity  $\alpha := \omega/r$  satisfies

$$(3) \quad \partial_t \alpha + (u \cdot \nabla) \alpha = 0,$$

which induces the conservation of all its  $L^p$  norms for every  $p \in [1, \infty]$ . Ukhovskii and Yudovich [22] took advantage of these conservation laws to prove the global existence for axisymmetric initial data with finite energy and satisfying in addition  $\omega^0 \in L^2 \cap L^\infty$  and  $\frac{\omega^0}{r} \in L^2 \cap L^\infty$ . In terms of Sobolev regularity these assumptions are satisfied if the velocity  $u_0 \in H^s$  with  $s > \frac{7}{2}$ . This is far from critical regularity of local existence theory  $s = \frac{5}{2}$ . The optimal result in Sobolev spaces is done by Shirota and Yanagisawa [20] who proved global existence in  $H^s$ , with  $s > \frac{5}{2}$ . In a recent work, R. Danchin [9] has weakened the Ukhovskii and Yudovich conditions. More precisely, he obtain the global existence and uniqueness for initial data  $\omega^0 \in L^{3,1} \cap L^\infty$  and  $\frac{\omega^0}{r} \in L^{3,1}$ . Recently, in [1] the first author and his collaborators proved the global existence to the system (E) for initial data  $u^0 \in B_{p,1}^{1+\frac{3}{p}}$  and  $\frac{\omega^0}{r} \in L^{3,1}$ .

In the same context (i.e geometric constraints), Dutrifoy was interested in this question and he was published several papers, in [10], he proved global existence to the incompressible Euler equations with a particular geometric structure, the focus is on so-called helicoidal solutions. In [9], Danchin proved too global existence for helicoidal initial data and the aim in this paper is to prescribe regularity conditions on the vorticity.

**Definition 1.1.** *Let  $k$  be a nonnegative real number. We say that a vector field  $u = u_r e_r + u_\theta e_\theta + u_z e_z$  is helicoidal if:*

- 1) *The components  $u_r, u_\theta$  et  $u_z$  of  $u$  are constant on helicoids  $z = z_0 + k\theta$  et  $r = r_0$ .*
- 2) *At every point of  $\mathbb{R}^3$  the vector field  $u$  is orthogonal to  $h := r e_\theta + k e_z$ .*

We note that the limit case  $k = 0$  corresponds to the definition of an axisymmetric vector field. The main characteristic of helical flows is the vorticity takes the following form:

$$k\omega = h\omega_z \quad \text{and} \quad \partial_t \omega_z + (u \cdot \nabla) \omega_z = 0$$

where  $\omega_z$  is the vertical component of the vorticity. Thus

$$\frac{|\omega(t, \psi(t, x))|}{\sqrt{k^2 + r^2(\psi(t, x))}} = \frac{|\omega(0, x)|}{\sqrt{k^2 + r^2}} \quad \text{where } \psi \text{ is the flow associated to velocity } u.$$

In this paper we shall not be interested in the dependence with regard to  $k$  quantities to be measured, and we shall thus suppose to simplify that  $k = 1$ . Our main result in this paper is concerning the unique solvability of (E) with the initial data helicoidal in the critical Besov spaces (for the definition see the next section). Here and in what follows, we shall always denote  $(1, x, y)f = (f, xf, yf)$ . More precisely we obtain the following result.

**Theorem 1.1.** *Let  $u^0$  be an helicoidal divergence free vector field with  $(1, x, y)u^0 \in L^2(\mathbb{R}^2 \times ]-\pi, \pi[)$ , such that its vorticity satisfies  $\omega^0 \in \dot{\mathcal{B}}_{2,1}^0(\mathbb{R}^2 \times ]-\pi, \pi[)$ ,  $(1, x, y)\omega^0 \in \dot{\mathcal{B}}_{\infty,1}^0(\mathbb{R}^2 \times ]-\pi, \pi[)$  and  $\omega_z^0 \in \dot{\mathcal{B}}_{1,1}^0(\mathbb{R}^2 \times ]-\pi, \pi[)$ . Then the system (E) has a unique global solution  $(1, x, y)u \in L_{loc}^\infty(\mathbb{R}_+; L^2)$  such that  $\omega \in L_{loc}^\infty(\mathbb{R}_+; \dot{\mathcal{B}}_{2,1}^0)$ ,  $(1, x, y)\omega \in L_{loc}^\infty(\mathbb{R}_+; \dot{\mathcal{B}}_{\infty,1}^0)$  and  $\omega_z \in L_{loc}^\infty(\mathbb{R}_+; \dot{\mathcal{B}}_{1,1}^0)$ . Moreover for every  $t \in \mathbb{R}_+$*

$$\|(1, x, y)u(t)\|_{L^2} \leq C_0 e^{C_0 t}$$

and

$$\|\omega_z(t)\|_{\dot{\mathcal{B}}_{1,1}^0} + \|\omega(t)\|_{\dot{\mathcal{B}}_{2,1}^0} + \|(1, x, y)\omega(t)\|_{\dot{\mathcal{B}}_{\infty,1}^0} \leq C_0 e^{\exp(C_0 t)}$$

where  $C_0$  depends on the norms of  $u^0$ .

**Remark 1.1.** According to [9],  $u^0 \in L^2(\mathbb{R}^2 \times ]-\pi, \pi[)$  can be replaced by  $\omega^0 \in L^{2,1}(\mathbb{R}^2 \times ]-\pi, \pi[)$ .

**Scheme of the proof and organization of the paper.** The main difficulty is the proof of Theorem 1.1 lies in the fact that when the initial data belongs to critical spaces, we can not use the Beale-Kato-Majda criterion. Thus, we owe controlled  $\|\nabla u\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)}$ , which is bounded by  $\sum_{n \in \mathbb{Z}} \|\omega_n\|_{\dot{B}_{\infty,1}^0(\mathbb{R}^2)}$  (where  $\omega_n$  is the Fourier coefficients of  $\omega$  see Lemma 3.1). For that we shall rewrite (1) (see Corollary 3.1)

$$\partial_t \omega + (\tilde{u} \cdot \nabla_h) \omega = \begin{pmatrix} -\omega_z \tilde{u}_2 \\ \omega_z \tilde{u}_1 \\ 0 \end{pmatrix},$$

where we denote

$$\tilde{u} = (u_1 + yu_3, u_2 - xu_3), \quad \nabla_h = (\partial_x, \partial_y) \quad \text{and} \quad \omega = (\omega_1, \omega_2, \omega_z) = (-y\omega_z, x\omega_z, \omega_z).$$

Motivated by [1, 13], for some  $n \in \mathbb{Z}$ , let  $\tilde{\omega}_n$  solves the following system

$$\begin{cases} \partial_t \tilde{\omega}_{1,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,n} = -\tilde{\omega}_{z,n} \tilde{u}_2, \\ \partial_t \tilde{\omega}_{2,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,n} = \tilde{\omega}_{z,n} \tilde{u}_1, \\ \partial_t \tilde{\omega}_{z,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,n} = 0, \\ \tilde{\omega}_n|_{t=0} = \tilde{\omega}_n(0) \end{cases}$$

where

$$\tilde{\omega}_n(0) = \begin{pmatrix} -y\omega_{n,z}^0 \\ x\omega_{n,z}^0 \\ \omega_{n,z}^0 \end{pmatrix}, \quad \text{and} \quad \omega_{n,z}^0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega_z^0 e^{-inz} dz = \partial_x u_{n,2}^0 - \partial_y u_{n,1}^0.$$

By Proposition 3.4, we deduce that  $\tilde{\omega}_n$  is the Fourier coefficients of  $\omega$ , i.e,  $\tilde{\omega}_n = \omega_n$ . Thus  $\omega = \sum_{n \in \mathbb{Z}} \omega_n e^{inz}$ . Finally to control  $\|\omega_n\|_{\dot{B}_{\infty,1}^0}$ , we will use a new approach similar to [13], which consists to linearize properly the Fourier of transport equation. For that, we will localize in frequency the initial data and denote by  $\tilde{\omega}_{q,n}$  the unique global vector-valued solution of the problem

$$\begin{cases} \partial_t \tilde{\omega}_{1,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,q,n} = -\tilde{\omega}_{z,q,n} \tilde{u}_2, \\ \partial_t \tilde{\omega}_{2,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,q,n} = \tilde{\omega}_{z,q,n} \tilde{u}_1, \\ \partial_t \tilde{\omega}_{z,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,q,n} = 0, \\ \tilde{\omega}_{q,n}|_{t=0} = \tilde{\omega}_{q,n}(0) \end{cases}$$

where

$$\tilde{\omega}_{q,n}(0) = \begin{pmatrix} -y\dot{\Delta}_q \omega_{n,z}^0 \\ x\dot{\Delta}_q \omega_{n,z}^0 \\ \dot{\Delta}_q \omega_{n,z}^0 \end{pmatrix}, \quad \text{and} \quad \dot{\Delta}_q \omega_{n,z}^0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \dot{\Delta}_q \omega_z^0 e^{-inz} dz = \partial_x \dot{\Delta}_q u_{n,2}^0 - \partial_y \dot{\Delta}_q u_{n,1}^0.$$

In the second section, we shall collect some basic facts on Littlewood-Paley analysis; then in section 3 is devoted to the study of some geometric properties of any solution to a vorticity equation model; finally in the last section, we prove Theorem 1.1.

**Notations:** Let  $A, B$  be two operators, we denote  $[A, B] = AB - BA$ , the commutator between  $A$  and  $B$ . For  $a \lesssim b$ , we mean that there is a uniform constant  $C$ , which may be different on different lines, such that  $a \leq Cb$ . For  $X$  a Banach space and  $I$  an interval of  $\mathbb{R}$ , we denote by  $\mathcal{C}(I; X)$  the set of continuous functions on  $I$  with values in  $X$ . For  $q \in [1, +\infty]$ , the notation  $L^q(I; X)$  stands for the set of measurable functions on  $I$  with values in  $X$ , such that  $t \mapsto \|f(t)\|_X$  belongs to  $L^q(I)$ . We always denote the Fourier transform of a function  $u$  by  $\hat{u}$  or  $\mathcal{F}(u)$ .

## 2. THE FUNCTIONAL TOOL BOX

The proof of Theorem 1.1 requires a dyadic decomposition of the Fourier variables, or Littlewood-Paley decomposition (see [2]). Let  $\varphi \in \mathcal{S}(\mathbb{R}^2)$  be smooth function supported in  $\mathcal{C} = \{\xi \in \mathbb{R}^2, \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$  such that

$$\sum_{q \in \mathbb{Z}} \varphi(2^{-q}\xi) = 1 \quad \text{for } \xi \neq 0.$$

For every  $u \in \mathcal{S}'(\mathbb{R}^2)$  one defines the homogeneous Littlewood-Paley operators by

$$\forall q \in \mathbb{Z}, \quad \dot{\Delta}_q u = \varphi(2^{-q}D)u \quad \text{and} \quad \dot{S}_q u = \sum_{j \leq q-1} \dot{\Delta}_j u.$$

We notice that these operators can be written as a convolution. For example for  $q \in \mathbb{Z}$ ,  $\dot{\Delta}_q u = 2^{2q} h(2^q \cdot) \star u$ , where  $h \in \mathcal{S}$  and  $\hat{h}(\xi) = \varphi(\xi)$ .

We have the formal decomposition

$$u = \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u, \quad \forall u \in \mathcal{S}'(\mathbb{R}^2)/\mathcal{P}[\mathbb{R}^2],$$

where  $\mathcal{P}[\mathbb{R}^2]$  is the set of polynomials (see [17]). Moreover, the Littlewood-Paley decomposition satisfies the property of almost orthogonality:

$$(4) \quad \dot{\Delta}_k \dot{\Delta}_q u \equiv 0 \quad \text{if } |k - q| \geq 2 \quad \text{and} \quad \dot{\Delta}_k (\dot{S}_{q-1} u \dot{\Delta}_q u) \equiv 0 \quad \text{if } |k - q| \geq 5.$$

We recall now the definition of homogeneous Besov type spaces from [2].

**Definition 2.1.** Let  $(p, r) \in [1, +\infty]^2$ ,  $s \in \mathbb{R}$  and  $u \in \mathcal{S}'(\mathbb{R}^2)$ , we set

$$\|u\|_{\dot{B}_{p,r}^s} = \left( 2^{qs} \|\dot{\Delta}_q u\|_{L^p} \right)_{\ell^r}.$$

- For  $s < \frac{2}{p}$  (or  $s = \frac{2}{p}$  if  $r = 1$ ), we define  $\dot{B}_{p,r}^s(\mathbb{R}^2) = \{u \in \mathcal{S}'(\mathbb{R}^2) \mid \|u\|_{\dot{B}_{p,r}^s} < \infty\}$ .
- If  $k \in \mathbb{N}$  and  $\frac{2}{p} + k \leq s < \frac{2}{p} + k + 1$  (or  $s = \frac{2}{p} + k + 1$  if  $r = 1$ ), then  $\dot{B}_{p,r}^s(\mathbb{R}^2)$  is defined as the subset of distributions  $u \in \mathcal{S}'(\mathbb{R}^2)$  such that  $\partial^\beta u \in \dot{B}_{p,r}^{s-k}(\mathbb{R}^2)$  whenever  $|\beta| = k$ .

**Remark 2.1.** (1) We point out that if  $s > 0$  then  $B_{p,r}^s = \dot{B}_{p,r}^s \cap L^p$  and

$$\|u\|_{B_{p,r}^s} \approx \|u\|_{\dot{B}_{p,r}^s} + \|u\|_{L^p}$$

with  $B_{p,r}^s$  being the non-homogeneous Besov space.

- (2) It is easy to verify that the homogeneous Besov space  $\dot{B}_{2,2}^s(\mathbb{R}^2)$  coincides with the classical homogeneous Sobolev space  $\dot{H}^s(\mathbb{R}^2)$  and  $\dot{B}_{\infty,\infty}^s(\mathbb{R}^2)$  coincides with the classical homogeneous Hölder space  $\dot{C}^s(\mathbb{R}^2)$  when  $s$  is not positive integer, in case  $s$  is a nonnegative integer,  $\dot{B}_{\infty,\infty}^s(\mathbb{R}^2)$  coincides with the classical homogeneous Zygmund space  $\dot{C}_*^s(\mathbb{R}^2)$ .
- (3) Let  $s \in \mathbb{R}, 1 \leq p, r \leq \infty$ , and  $u \in \mathcal{S}'(\mathbb{R}^2)$ . Then  $u$  belongs to  $\dot{B}_{p,r}^s(\mathbb{R}^2)$  if and only if there exists  $\{c_{j,r}\}_{j \in \mathbb{Z}}$  such that  $\|c_{j,r}\|_{\ell^r} = 1$  and

$$\|\dot{\Delta}_j u\|_{L^p} \leq C c_{j,r} 2^{-js} \|u\|_{\dot{B}_{p,r}^s} \quad \text{for all } j \in \mathbb{Z}.$$

For the convenience of the reader, we recall some basic facts on Littlewood-Paley theory, one may check [2] for more details.

**Lemma 2.1.** *Let  $\mathcal{B}$  be a ball and  $\mathcal{C}$  an annulus of  $\mathbb{R}^N$ . A constant  $C$  exists so that for any positive real number  $\delta$ , any non-negative integer  $k$ , any smooth homogeneous function  $\sigma$  of degree  $m$ , and any couple of real numbers  $(a, b)$  with  $b \geq a \geq 1$ , there hold*

$$\begin{aligned} \text{Supp } \hat{u} \subset \delta\mathcal{B} &\Rightarrow \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^b} \leq C^{k+1} \delta^{k+N(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}, \\ \text{Supp } \hat{u} \subset \delta\mathcal{C} &\Rightarrow C^{-1-k} \delta^k \|u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha u\|_{L^a} \leq C^{1+k} \delta^k \|u\|_{L^a}, \\ \text{Supp } \hat{u} \subset \delta\mathcal{C} &\Rightarrow \|\sigma(D)u\|_{L^b} \leq C_{\sigma,m} \delta^{m+N(\frac{1}{a}-\frac{1}{b})} \|u\|_{L^a}. \end{aligned}$$

In what follows, we shall frequently use Bony's decomposition [5] in the both homogeneous and inhomogeneous context:

$$uv = \dot{T}_u v + \dot{R}(u, v) = \dot{T}_u v + \dot{T}_v u + \dot{\mathcal{R}}(u, v)$$

where

$$\begin{aligned} \dot{T}_u v &= \sum_{q \in \mathbb{Z}} \dot{S}_{q-1}^h u \dot{\Delta}_q^h v, & \dot{R}(u, v) &= \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \dot{S}_{q+2} v, \\ \dot{\mathcal{R}}(u, v) &= \sum_{q \in \mathbb{Z}} \dot{\Delta}_q u \tilde{\Delta}_q v \quad \text{with} \quad \tilde{\Delta}_q v = \sum_{|q'-q| \leq 1} \dot{\Delta}_{q'} v. \end{aligned}$$

**Definition 2.2.** *Let  $u$  be a mean free function in  $\mathcal{S}'(\mathbb{R}^2 \times ]-\pi, \pi[)$ ,  $2\pi$ -periodic with respect the third variable,  $(p, r) \in [1, +\infty]^2$  and  $s \in \mathbb{R}$  be given real numbers. Then  $u$  belongs to the Besov space  $\dot{\mathcal{B}}_{p,r}^s$  if and only if*

$$\|u\|_{\dot{\mathcal{B}}_{p,r}^s} = \sum_{n \in \mathbb{Z}} \|u_n\|_{\dot{B}_{p,r}^s} < +\infty$$

where  $u_n$  is the Fourier coefficients are computed as follows

$$u_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(\cdot, \cdot, z) e^{-inz} dz.$$

**Remark 2.2.** *Let  $(p, r) \in [1, +\infty]^2$  and  $s \in \mathbb{R}$ , then for all  $u \in \mathcal{S}'(\mathbb{R}^2 \times ]-\pi, \pi[)$ ,  $2\pi$ -periodic with regard to the third variable, we have*

$$\|u\|_{\dot{B}_{p,r}^s(\mathbb{R}^2)} \lesssim \|u\|_{\dot{\mathcal{B}}_{p,r}^s}.$$

### 3. GEOMETRIC PROPERTIES OF THE VORTICITY

**Proposition 3.1.** *Let  $u = (u_1, u_2, u_z)$  be a smooth helicoidal vector field. Then the vector  $\omega = \nabla \times u = (\omega_1, \omega_2, \omega_z)$  satisfies for every  $(x_1, x_2, z) \in \mathbb{R}^3$ ,*

$$x_1 \omega_1(x_1, x_2, z) + x_2 \omega_2(x_1, x_2, z) = 0.$$

*Proof.* In the cylindrical coordinate system, we have

$$(5) \quad \omega = \nabla \times u = \begin{pmatrix} \frac{1}{r} \partial_\theta u_z - \partial_z u_\theta \\ \partial_z u_r - \partial_r u_z \\ \frac{u_\theta}{r} + \partial_r u_\theta - \frac{1}{r} \partial_\theta u_r \end{pmatrix},$$

and the second point of the Definition 1.1, implies that

$$\omega = r \omega_z e_\theta + \omega_z e_z.$$

Then

$$\omega^1(x_1, x_2, z) = -x_2 \omega_z$$

and

$$\omega^2(x_1, x_2, z) = x_1 \omega_z.$$

Therefore

$$x_1 \omega_1(x_1, x_2, z) + x_2 \omega_2(x_1, x_2, z) = 0.$$

This achieves the proof.  $\square$

**Proposition 3.2.** *Let  $u$  be an helicoidal vector field. Then*

$$\left| \partial_r \left( \frac{u_\theta}{r} \right) \right| + \left| \partial_z \left( \frac{u_\theta}{r} \right) \right| \lesssim |\partial_x^2 u| + |\partial_y^2 u| + |\partial_{xy}^2 u| + |\nabla u|.$$

*Proof.* According to the inequality (5), we have

$$\frac{u_\theta}{r} = \omega_z - \partial_r u_\theta + \frac{1}{r} \partial_\theta u_r$$

where  $\omega_z$  is the vertical component of  $\text{rot } u$ . One has

$$\partial_r = \cos(\theta) \partial_x + \sin(\theta) \partial_y$$

and

$$\frac{1}{r} \partial_\theta = -\sin(\theta) \partial_x + \cos(\theta) \partial_y$$

it follows that

$$\partial_r \left( \frac{1}{r} \partial_\theta \right) = -\sin(\theta) \cos(\theta) \partial_x^2 + \sin(\theta) \cos(\theta) \partial_y^2 + (\cos^2(\theta) - \sin^2(\theta)) \partial_{xy}^2$$

and

$$\partial_r^2 = \cos^2(\theta) \partial_x^2 + \cos^2(\theta) \partial_y^2 + 2 \cos(\theta) \cos(\theta) \partial_{xy}^2.$$

Thus we find

$$\left| \partial_r \left( \frac{u_\theta}{r} \right) \right| \lesssim |\partial_x^2 u| + |\partial_y^2 u| + |\partial_{xy}^2 u|.$$

Since  $u$  is helicoidal, then

$$\partial_z \left( \frac{u_\theta}{r} \right) = -\frac{1}{r} \partial_\theta u_\theta$$

thus

$$\left| \partial_z \left( \frac{u_\theta}{r} \right) \right| \lesssim |\nabla u|.$$

This finishes the proof of Proposition.  $\square$

• The last part of this section is dedicated to the study of a vorticity equation type in which no relations between the vector field  $u$  and the solution  $\Omega$  are supposed. More precisely, we consider

$$(6) \quad \begin{cases} \partial_t \Omega + u \cdot \nabla \Omega = \Omega \cdot \nabla u, \\ \text{div } u = 0 \\ \Omega|_{t=0} = \Omega^0. \end{cases}$$

**Proposition 3.3.** *Let  $u$  be a divergence free and helicoidal vector field such that  $\nabla u$  and  $\nabla^2 u$  belonging to  $L_{loc}^1(\mathbb{R}_+; L^\infty(\mathbb{R}^3))$  and  $\Omega$  the unique global solution of (6) with smooth initial data  $\Omega^0$ . Then, the following properties hold.*

- i) *If  $\text{div } \Omega^0 = 0$ , then  $\text{div } \Omega(t) = 0$ , for every  $t \in \mathbb{R}_+$ .*
- ii) *If  $\Omega^0 = r \Omega_z^0 e_\theta + \Omega_z^0 e_z$ , then we have*

$$\Omega(t) = r \Omega_z(t) e_\theta + \Omega_z(t) e_z, \quad \forall t \in \mathbb{R}_+.$$

Consequently,  $\Omega(t, x_1, 0, z) = \Omega(t, 0, x_2, z) = 0$  and

$$\partial_t \Omega + (u \cdot \nabla) \Omega = \Omega_z (u_r e_\theta - u_\theta e_r).$$

*Proof.* First, we notice that the existence and uniqueness of global solution can be done in classical way. Indeed, let  $\psi$  the flow of the velocity  $u$ ,

$$\psi(t, x) = x + \int_0^t u(\tau, \psi(\tau, x)) d\tau.$$

Since  $u \in L_{loc}^1(\mathbb{R}_+; Lip(\mathbb{R}^3))$  then it follows from the ODE theory that the function  $\psi$  is uniquely and globally defined.

Let  $\tilde{\Omega}(t, x) := \Omega(t, \psi(t, x))$  and  $A(t, x)$  the matrix such that  $A(t, \psi^{-1}(t, x)) = (\partial_j u_i)_{1 \leq i, j \leq 3}$ .

It's clear that

$$\partial_t \tilde{\Omega} = A(t, x) \tilde{\Omega}.$$

From Cauchy–Lipschitz theorem this last equation has a unique global solution, and the system (6) too.

i) We apply the divergence operator to the equation (6), leading under the assumption  $\operatorname{div} u = 0$ , to

$$\partial_t \operatorname{div} \Omega + u \cdot \nabla \operatorname{div} \Omega = 0.$$

Then, the quantity  $\operatorname{div} \Omega$  is transported by the flow and consequently the incompressibility of  $\Omega$  remains true for every time.

ii) We have

$$(u \cdot \nabla \Omega) \cdot e_r = u \cdot \nabla \Omega_r - \frac{1}{r} u_\theta \Omega_\theta$$

and

$$(\Omega \cdot \nabla u) \cdot e_r = \Omega \cdot \nabla u_r - \frac{1}{r} \Omega_\theta u_\theta$$

then the component  $\Omega_r$ , verifies

$$(7) \quad \partial_t \Omega_r + u \cdot \nabla \Omega_r = \Omega \cdot \nabla u_r = \Omega_r \partial_r u_r + \left( \Omega_z - \frac{\Omega_\theta}{r} \right) \partial_z u_r.$$

From the maximum principle we deduce

$$\|\Omega_r(t)\|_{L^\infty} \leq \int_0^t (\|\Omega_r(\tau)\|_{L^\infty} + \|(\Omega_z - \frac{\Omega_\theta}{r})(\tau)\|_{L^\infty}) \|\nabla u(\tau)\|_{L^\infty} d\tau.$$

The component  $\Omega_\theta$  satisfies the following equation

$$\partial_t \Omega_\theta + u \cdot \nabla \Omega_\theta = \Omega_r \partial_r u_\theta + \left( \Omega_z - \frac{\Omega_\theta}{r} \right) \partial_z u_\theta + \frac{1}{r} \Omega_\theta u_r - \frac{1}{r} \Omega_r u_\theta,$$

therefore

$$\partial_t \frac{\Omega_\theta}{r} + u \cdot \nabla \left( \frac{\Omega_\theta}{r} \right) = \Omega_r \partial_r \left( \frac{u_\theta}{r} \right) + \left( \Omega_z - \frac{\Omega_\theta}{r} \right) \partial_z \left( \frac{u_\theta}{r} \right).$$

Since the component  $\Omega_z$  satisfies the following equation

$$\partial_t \Omega_z + u \cdot \nabla \Omega_z = \Omega_r \partial_r u_z + \left( \Omega_z - \frac{\Omega_\theta}{r} \right) \partial_z u_z,$$

then

$$\partial_t \left( \Omega_z - \frac{\Omega_\theta}{r} \right) + u \cdot \nabla \left( \Omega_z - \frac{\Omega_\theta}{r} \right) = \Omega_r \partial_r \left( u_z - \frac{u_\theta}{r} \right) + \left( \Omega_z - \frac{\Omega_\theta}{r} \right) \partial_z \left( u_z - \frac{u_\theta}{r} \right).$$

Thus from the maximum principle and Proposition 3.2

$$\begin{aligned} & \left\| \left( \Omega_z - \frac{\Omega_\theta}{r} \right)(t) \right\|_{L^\infty} \\ & \leq \int_0^t (\|\Omega_r(\tau)\|_{L^\infty} + \|(\Omega_z - \frac{\Omega_\theta}{r})(\tau)\|_{L^\infty}) (\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty}) d\tau. \end{aligned}$$

Then

$$\begin{aligned} & \|\Omega_r(t)\|_{L^\infty} + \left\| \left( \Omega_z - \frac{\Omega_\theta}{r} \right)(t) \right\|_{L^\infty} \\ & \lesssim \int_0^t \left( \|\Omega_r(\tau)\|_{L^\infty} + \left\| \left( \Omega_z - \frac{\Omega_\theta}{r} \right)(\tau) \right\|_{L^\infty} \right) (\|\nabla u\|_{L^\infty} + \|\nabla^2 u\|_{L^\infty}) d\tau. \end{aligned}$$

Applying Gronwall inequality gives

$$\Omega_r(t) = 0 \quad \text{and} \quad r\Omega_z(t) = \Omega_\theta(t), \quad \forall t \in \mathbb{R}_+.$$

Combining the previous estimate with the fact that  $u$  is helicoidal, we obtain

$$\begin{aligned} \Omega \cdot \nabla u &= \Omega_r \partial_r u + \frac{1}{r} \Omega_\theta \partial_\theta u + \Omega_z \partial_z u \\ &= \Omega_z (\partial_\theta + \partial_z) (u_r e_r + u_\theta e_\theta + u_z e_z) \\ &= \Omega_z (u_r e_\theta - u_\theta e_r). \end{aligned}$$

Which ends the proof of this Proposition.  $\square$

An immediate corollary of the above Proposition gives

**Corollary 3.1.** *Let  $u$  be an helicoidal divergence free vector field solution of the Euler equations, then  $\omega = \text{rot } u$  verifies*

$$\begin{cases} \partial_t \omega_1 + (u_1 + y u_3) \partial_x \omega_1 + (u_2 - x u_3) \partial_y \omega_1 = \omega_2 u_3 - \omega_z u_2, \\ \partial_t \omega_2 + (u_1 + y u_3) \partial_x \omega_2 + (u_2 - x u_3) \partial_y \omega_2 = \omega_z u_1 - \omega_1 u_3, \\ \partial_t \omega_z + (u_1 + y u_3) \partial_x \omega_z + (u_2 - x u_3) \partial_y \omega_z = 0, \end{cases}$$

with

$$\partial_x (u_1 + y u_3) + \partial_y (u_2 - x u_3) = 0.$$

*Proof.* By the above Proposition, we have

$$\omega_1 = -y \omega_z \quad \text{and} \quad \omega_2 = x \omega_z$$

with  $\omega_z$  verify

$$\partial_t \omega_z + (u_1 \partial_x + u_2 \partial_y) \omega_z + u_3 \partial_z \omega_z = 0.$$

While since

$$\text{div } \omega = \partial_x \omega_1 + \partial_y \omega_2 + \partial_z \omega_z = 0,$$

we have

$$\partial_z \omega_z = -\partial_x \omega_1 - \partial_y \omega_2 = y \partial_x \omega_z - x \partial_y \omega_z,$$

thus

$$\partial_t \omega_3 + (u_1 + y u_3) \partial_x \omega_z + (u_2 - x u_3) \partial_y \omega_z = 0.$$

As  $\omega_1 = -y \omega_z$  and  $\omega_2 = x \omega_z$ , then

$$\partial_t \omega_1 + (u_1 + y u_3) \partial_x \omega_1 + (u_2 - x u_3) \partial_y \omega_1 = \omega_z (x u_3 - u_2) = \omega_2 u_3 - \omega_z u_2$$

and

$$\partial_t \omega_2 + (u_1 + y u_3) \partial_x \omega_2 + (u_2 - x u_3) \partial_y \omega_2 = \omega_z (u_1 + y u_3) = \omega_z u_1 - \omega_1 u_3.$$

Concerning the second point, we have

$$\begin{aligned} \partial_x (u_1 + y u_3) + \partial_y (u_2 - x u_3) &= \partial_x u_1 + \partial_y u_2 + y \partial_x u_3 - x \partial_y u_3 = \partial_x u_1 + \partial_y u_2 - \partial_\theta u_3 \\ &= \partial_x u_1 + \partial_y u_2 + \partial_z u_3 \\ &= 0, \end{aligned}$$

and we are done.  $\square$

To prove our theorem, we need the following proposition which describes the distribution of the Fourier coefficients to transport equation.



**Proposition 3.4.** *Under the assumptions in Corollary 3.1. If  $\partial_z \omega^0 = 0$  with  $\omega^0$  is the initial data, then*

$$\partial_z \omega = 0.$$

*Proof.* By taking  $\partial_z$  to the  $\omega_z$  equation, we obtain

$$\partial_t \partial_z \omega_z + (u_1 + yu_3) \partial_x \partial_z \omega_z + (u_2 - xu_3) \partial_y \partial_z \omega_z = -\partial_z(u_1 + yu_3) \partial_x \omega_z - \partial_z(u_2 - xu_3) \partial_y \omega_z.$$

The fact that  $\operatorname{div} u = 0$ ,  $\partial_x(v_1 + yv_3) + \partial_y(v_2 - xv_3) = 0$ ,  $\omega = (-y\omega_z, x\omega_z, \omega_z)$  and  $u_3 = yu_1 - xu_2$ , leads to

$$\begin{aligned} \partial_z(u_1 + yu_3) &= x\omega_z + \partial_x u_3 - y\partial_x u_1 - y\partial_y u_2 \\ &= x\partial_x u_2 - x\partial_y u_1 + \partial_x(yu_1 - xu_2) - y\partial_x u_1 - y\partial_y u_2 \\ &= x\partial_x u_2 - \partial_y(xu_1) + y\partial_x u_1 - \partial_x(xu_2) - y\partial_x u_1 - y\partial_y u_2 \\ &= -\partial_y(ru_r). \end{aligned}$$

A similar procedure gives rise to

$$\partial_z(u_2 - xu_3) = \partial_x(ru_r).$$

Hence we obtain

$$\begin{aligned} \partial_z(u_1 + yu_3) \partial_x \omega_z + \partial_z(u_2 - xu_3) \partial_y \omega_z &= \partial_y(-ru_r) \partial_x \omega_z + \partial_x(ru_r) \partial_y \omega_z \\ &= \partial_x[\omega_z \partial_y(-ru_r)] + \partial_y[\omega_z \partial_x(ru_r)] \\ &= \partial_x[-ru_r \partial_y \omega_z] + \partial_y[ru_r \partial_x \omega_z] \\ &= \partial_x(-ru_r) \partial_y \omega_z + \partial_y(ru_r) \partial_x \omega_z, \end{aligned}$$

from which, we infer

$$\partial_t \partial_z \omega_z + (u_1 + yu_3) \partial_x \partial_z \omega_z + (u_2 - xu_3) \partial_y \partial_z \omega_z = 0.$$

Applying maximum principle and Gronwall's inequality, we deduce

$$\partial_z \omega_z = 0,$$

and as a consequence  $\partial_z \omega = 0$ , because  $\omega = (-y\omega_z, x\omega_z, \omega_z)$ . This completes the proof of the proposition.  $\square$

To prove Theorem 1.1, we need the following two technical lemmas:

**Lemma 3.1.** *Let  $v = (v^1, v^2, v^3)$  be a divergence free vector field  $2\pi$ -periodic with respect the third variable, then*

$$\|\nabla_h v\|_{\dot{\mathcal{B}}_{\infty,1}^0} \lesssim \|\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0},$$

$$\|\dot{\Delta}_j \nabla_h v\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim 2^{\frac{j}{2}} \|\dot{\Delta}_j \Omega\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)}, \quad j \geq 0$$

and

$$\|\dot{S}_0 v(x_h, z)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|\Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)},$$

with

$$\nabla_h = (\partial_x, \partial_y), \quad \text{and} \quad \Omega = \operatorname{curl} v.$$

*Proof.* We have

$$v(x, y, z) = \sum_{n \in \mathbb{Z}} v_n(x, y) e^{inz}$$

and

$$\Omega(x, y, z) = \sum_{n \in \mathbb{Z}} \Omega_n(x, y) e^{inz},$$

where  $v_n$  is the Fourier coefficients are computed as follows

$$v_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} v(\cdot, \cdot, z) e^{-inz} dz.$$

Using

$$(\Delta_h + \partial_z^2)v = -\operatorname{curl} \Omega,$$

we find that

$$(-n^2 + \Delta_h)v_n^3 = \partial_y \Omega_n^1 - \partial_x \Omega_n^2.$$

Localizes it in horizontal Fourier

$$\begin{aligned} \mathcal{F}^h(\dot{\Delta}_j v_n^3)(\xi_h) &= \frac{\xi_1}{n^2 + |\xi_h|^2} \mathcal{F}^h(\dot{\Delta}_j \Omega_n^2)(\xi_h) - \frac{\xi_2}{n^2 + |\xi_h|^2} \mathcal{F}^h(\dot{\Delta}_j \Omega_n^1)(\xi_h) \\ &= \frac{\xi_1}{n^2 + |\xi_h|^2} \tilde{\varphi}(2^{-j} \xi_h) \mathcal{F}^h(\dot{\Delta}_j \Omega_n^2)(\xi_h) - \frac{\xi_2}{n^2 + |\xi_h|^2} \tilde{\varphi}(2^{-j} \xi_h) \mathcal{F}^h(\dot{\Delta}_j \Omega_n^1)(\xi_h) \end{aligned}$$

with  $\tilde{\varphi} \in \mathcal{S}(\mathbb{R}^2)$  is a smooth function supported in  $\mathcal{C} = \{\xi \in \mathbb{R}^2, 0 < R_1 \leq |\xi| \leq R_2\}$  such that  $\tilde{\varphi} = 1$  on support of  $\varphi$ , thus

$$\mathcal{F}^h(\nabla_h \dot{\Delta}_j v_n^3)(\xi_h) = \frac{\xi_h \xi_1}{n^2 + |\xi_h|^2} \tilde{\varphi}(2^{-j} \xi_h) \mathcal{F}^h(\dot{\Delta}_j \Omega_n^2)(\xi_h) - \frac{\xi_h \xi_2}{n^2 + |\xi_h|^2} \tilde{\varphi}(2^{-j} \xi_h) \mathcal{F}^h(\dot{\Delta}_j \Omega_n^1)(\xi_h).$$

Let

$$\mathcal{F}^h(K_n^i)(\xi_h) = \frac{\xi_h \xi_i}{n^2 + |\xi_h|^2} \tilde{\varphi}(2^{-j} \xi_h) \quad \text{for } i = 1, 2,$$

then

$$\|K_n^i\|_{L^1} \lesssim \frac{2^{2j}}{n^2 + 2^{2j}}.$$

We thus obtain

$$\|\nabla_h \dot{\Delta}_j v_n^3\|_{L^\infty(\mathbb{R}^2)} \lesssim (\|K_n^1\|_{L^1} + \|K_n^2\|_{L^1}) \|\dot{\Delta}_j \Omega_n\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\dot{\Delta}_j \Omega_n\|_{L^\infty(\mathbb{R}^2)}.$$

Therefore

$$\|\nabla_h v^3\|_{\dot{\mathcal{B}}_{\infty,1}^0(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim \sum_{n,j} \|\dot{\Delta}_j \Omega_n\|_{L^\infty(\mathbb{R}^2)} = \|\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0}.$$

A similar argument gives the same estimate for  $\|\nabla_h v_i\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)}$  for  $i = 1, 2$ .

For the second inequality, we have

$$(8) \quad \sum_{n \in \mathbb{Z}} \frac{n^2}{(n^2 + \lambda^2)^2} \lesssim \lambda^{-1}, \quad \sum_{n \in \mathbb{Z}} \frac{1}{(n^2 + \lambda^2)^2} \lesssim \lambda^{-3} \quad \forall \lambda \geq 1$$

and

$$\sum_{n \in \mathbb{Z}} |\Omega_n|^2 = \|\Omega\|_{L^2(]-\pi, \pi[)}^2.$$

It follows that for  $j \geq 0$

$$\|\dot{\nabla}_h \Delta_j v\|_{L^\infty} \lesssim \left\{ \left\| \left( \sum_{n \in \mathbb{Z}} (K_n^i)^2 \right)^{\frac{1}{2}} \right\|_{L^1} + \left\| \left( \sum_{n \in \mathbb{Z}} (\kappa_n)^2 \right)^{\frac{1}{2}} \right\|_{L^1} \right\} \left\| \left( \sum_{n \in \mathbb{Z}} |\dot{\Delta}_j \Omega_n|^2 \right)^{\frac{1}{2}} \right\|_{L^\infty},$$

with

$$\mathcal{F}^h \kappa_n(\xi_h) = \frac{n \xi_h}{n^2 + |\xi_h|^2} \tilde{\varphi}(2^{-j} \xi_h).$$

When  $|x_h| \geq 1$ , we obtained thanks to stationary phase Theorem

$$|K_n^i(x_h)| + |\kappa_n(x_h)| \lesssim \frac{|n|}{n^2 + 2^{2j}} \frac{1}{|x_h|^3}$$

and for  $|x_h| \leq 1$ , we have

$$|K_n^i(x_h)| + |\kappa_n(x_h)| \lesssim \frac{2^j |n|}{n^2 + 2^{2j}} + \frac{2^{2j}}{n^2 + 2^{2j}}.$$

Finally thanks to (8), we have

$$\|\dot{\Delta}_j \nabla_h v\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim 2^{\frac{1}{2}j} \|\dot{\Delta}_j \Omega\|_{(L^\infty(\mathbb{R}^2); L^2(]-\pi, \pi[))} \lesssim 2^{\frac{1}{2}j} \|\Omega\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)}.$$

For the second inequality, we use the fact that

$$\begin{aligned} \|\dot{S}_0 v\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} &\lesssim \sum_{n \in \mathbb{Z}, q \leq 0} \|\dot{\Delta}_q v_n\|_{L^\infty(\mathbb{R}^2)} \\ &\lesssim \sum_{n \in \mathbb{Z}, q \leq 0} 2^q \|\dot{\Delta}_q v_n\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \sum_{q \leq 0} 2^q \|\dot{\Delta}_q v_0\|_{L^2(\mathbb{R}^2)} + \sum_{n \in \mathbb{Z}^*, q \leq 0} 2^q \frac{1}{n} \|\dot{\Delta}_q v_n\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \sum_{q \leq 0} 2^q \|\partial_z \dot{\Delta}_q v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)}, \end{aligned}$$

as

$$\partial_z v^1 = \Omega^2 + \partial_x v^3, \quad \partial_z v^2 = -\Omega^1 + \partial_y v^3 \quad \text{and} \quad \partial_z v^3 = -\partial_x v^1 - \partial_y v^2.$$

Therefore by virtue of Bernstein's inequality, we obtain

$$\begin{aligned} \|\dot{S}_0 v\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} &\lesssim \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|\Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \sum_{q \leq 0} 2^q \|\nabla_h \dot{\Delta}_q v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} \\ &\lesssim \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|\Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)}. \end{aligned}$$

This gives the desired result.  $\square$

**Remark 3.1.** As

$$\partial_z v^1 = \Omega^2 + \partial_x v^3, \quad \partial_z v^2 = -\Omega^1 + \partial_y v^3 \quad \text{and} \quad \partial_z v^3 = -\partial_x v^1 - \partial_y v^2,$$

then

$$\|\nabla v\|_{\dot{\mathcal{B}}_{\infty,1}^0} \lesssim \|\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0}.$$

Following a same approach, we obtain

$$\|\nabla v\|_{\dot{\mathcal{B}}_{2,1}^0} \lesssim \|\Omega\|_{\dot{\mathcal{B}}_{2,1}^0}.$$

**Lemma 3.2.** Let  $v$  be divergence free vector field  $2\pi$ -periodic with respect the third variable, then

$$\|\nabla_h(xv)\|_{\dot{\mathcal{B}}_{\infty,1}^0} + \|\nabla_h(yv)\|_{\dot{\mathcal{B}}_{\infty,1}^0} \lesssim \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|(x, y)\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0} + \|\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0},$$

$$\|\dot{\Delta}_j \nabla_h((x, y)v)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim 2^{\frac{j}{2}} (\|\dot{\Delta}_j((x, y)\Omega)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|\dot{\Delta}_j \Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)}), \quad j \geq 0$$

and

$$\|\dot{S}_0(x, y)v\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim \|(x, y)v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|(x, y)\Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)},$$

with

$$\Omega = \text{curl } v.$$

*Proof.* We have

$$-\Delta(xv) = \text{curl}(\text{curl}(xv)) - \nabla v^1 = \text{curl}(x\Omega) + \begin{pmatrix} \partial_y v^2 - \partial_x v^1 \\ -\partial_x v^2 - \partial_y v^1 \\ -\partial_x v^3 - \partial_z v^1 \end{pmatrix}$$

and  $xv$  is  $2\pi$ -periodic with respect the third variable. Then we deduce from Lemma 3.1 that

$$\|\nabla_h(-\Delta)^{-1} \text{curl}(x\Omega)\|_{\dot{\mathcal{B}}_{\infty,1}^0} \lesssim \|x\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0}.$$

For the second terme, we write from the definition of  $\Omega$

$$\partial_z v^1 = \Omega^2 + \partial_x v^3,$$

$$\sum_{n \in \mathbb{Z}} \frac{1}{n^2 + 2^{2j}} \lesssim \begin{cases} 2^{-j}, & \text{if } j \geq 0 \\ 2^{-2j}, & \text{if } j \leq 0 \end{cases}$$

and

$$\|\dot{\Delta}_j v_n^3\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\dot{\Delta}_j v^3\|_{L^2(\cdot, -\pi, \pi[, L^\infty(\mathbb{R}^2))},$$

then

$$\begin{aligned} \|\nabla_h(-\Delta)^{-1} \partial_x v^3\|_{\dot{\mathcal{B}}_{\infty,1}^0} &\lesssim \sum_{j \leq 0} \|\dot{\Delta}_j v^3\|_{L^2(\cdot, -\pi, \pi[, L^\infty(\mathbb{R}^2))} + \sum_{j \geq 0} \|\dot{\Delta}_j \nabla_h v^3\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \\ &\lesssim \sum_{j \leq 0} 2^j \|\dot{\Delta}_j v^3\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \sum_{j \geq 0} \|\dot{\Delta}_j \nabla_h v^3\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \\ &\lesssim \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0} \end{aligned}$$

and

$$\|\nabla_h(-\Delta)^{-1} \Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0} \lesssim \|\Omega\|_{\dot{\mathcal{B}}_{\infty,1}^0}.$$

For the last inequality, we deduce by Lemma 3.1 and Bernstein inequality

$$\|\dot{\Delta}_j \nabla_h(-\Delta)^{-1} \text{curl}(x\Omega)\|_{L^\infty} \lesssim 2^{\frac{1}{2}j} \|\dot{\Delta}_j(x\Omega)\|_{L^\infty}$$

and

$$\|\dot{\Delta}_j \nabla_h(-\Delta)^{-1} \nabla v\|_{L^\infty} \lesssim 2^{\frac{1}{2}j} \|\dot{\Delta}_j \Omega\|_{L^2}$$

Finally for the last inequality, let us use the fact that

$$\begin{aligned} \|\dot{S}_0(xv)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} &\lesssim \sum_{n \in \mathbb{Z}, q \leq 0} \|\dot{\Delta}_q(xv_n)\|_{L^\infty(\mathbb{R}^2)} \\ &\lesssim \sum_{n \in \mathbb{Z}, q \leq 0} 2^q \|\dot{\Delta}_q(xv_n)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \sum_{q \leq 0} 2^q \|\dot{\Delta}_q(xv_0)\|_{L^2(\mathbb{R}^2)} + \sum_{n \in \mathbb{Z}^*, q \leq 0} 2^q \frac{1}{n} \|\dot{\Delta}_q(xv_n)\|_{L^2(\mathbb{R}^2)} \\ &\lesssim \|xv\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \sum_{q \leq 0} 2^q \|\partial_z \dot{\Delta}_q(xv)\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)}, \end{aligned}$$

as

$$\begin{aligned} x\partial_z v^1 &= x\Omega^2 + \partial_x(xv^3) - v^3, & x\partial_z v^2 &= -x\Omega^1 + \partial_y(xv^3) \\ \text{and } x\partial_z u_3 &= -\partial_x(xv^1) - \partial_y(xv^2) + v^1, \end{aligned}$$

Therefore by virtue of Bernstein inequality, we obtain

$$\begin{aligned} \|\dot{S}_0(xv)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} &\lesssim \|xv\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|x\Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} \\ &\quad + \sum_{q \leq 0} 2^q \|\nabla_h \dot{\Delta}_q(xv)\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} \\ &\lesssim \|xv\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|v\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|x\Omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)}. \end{aligned}$$

Similar for  $yv$ . This achieves the proof of the Lemma.  $\square$

## 4. PROOF OF THEOREM 1.1

4.1. **Some a priori estimates.** According to [9], we deduce the following proposition.

**Proposition 4.1.** *Let  $u$  be an helicoidal solution of (E), then we have for every  $t \in \mathbb{R}_+$ ,*

$$(9) \quad \|u(t)\|_{L^\infty} + \|\omega(t)\|_{L^\infty} \leq C(\|u^0\|_{L^2} + \|\omega^0\|_{L^\infty \cap L^2})e^{Ct\|\omega_z^0\|_{L^\infty \cap L^2}}$$

and

$$\|(xu, yu)(t)\|_{L^\infty} + \|(x\omega, y\omega)(t)\|_{L^\infty} \leq C(\|(1, x, y)u^0\|_{L^2} + \|(1, x, y)\omega^0\|_{L^\infty \cap L^2})e^{Ct\|\omega_z^0\|_{L^\infty \cap L^2}}.$$

*Proof.* Since  $\omega$ , satisfies the following equation

$$\partial_t \omega + (u \cdot \nabla) \omega = \omega_z(u_r e_\theta - u_\theta e_r),$$

thus, from the maximum principle we obtain

$$\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \int_0^t \|u(\tau)\|_{L^\infty} \|\omega_z(\tau)\|_{L^p} d\tau \quad \forall p \in [1, \infty].$$

Since  $\omega_z$  satisfies the transport equation, we have

$$\|\omega_z(t)\|_{L^p} \leq \|\omega_z^0\|_{L^p},$$

then

$$\|\omega(t)\|_{L^p} \leq \|\omega^0\|_{L^p} + \|\omega_z^0\|_{L^p} \int_0^t \|u(\tau)\|_{L^\infty} d\tau.$$

To estimate the  $L^\infty$  norm of the velocity, we use an argument of P. Serfati [19] and Lemma 3.1

$$\begin{aligned} \|u(t)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} &\leq \|\dot{S}_0 u\|_{L^\infty} + \sum_{q \geq 0} \|\dot{\Delta}_q u\|_{L^\infty} \\ &\lesssim \|u\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|\omega\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \sum_{q \geq 0} \|\dot{\Delta}_q u\|_{L^\infty}. \end{aligned}$$

By Bernstein inequality and Lemma 3.1, we deduce <sup>1</sup>

$$\sum_{q \geq 0} \|\dot{\Delta}_q u\|_{L^\infty} \lesssim \|\omega\|_{L^\infty}.$$

Consequently, we obtain

$$\|u(t)\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim \|u^0\|_{L^2} + \|\omega^0\|_{L^\infty \cap L^2} + \|\omega_z^0\|_{L^\infty \cap L^2} \int_0^t \|u(\tau)\|_{L^\infty} d\tau.$$

Using Gronwall's inequality, we have

$$\|u(t)\|_{L^\infty} \leq C(\|u^0\|_{L^2} + \|\omega^0\|_{L^\infty \cap L^2})e^{Ct\|\omega_z^0\|_{L^\infty \cap L^2}}.$$

By maximum principle, Gronwall's inequality and inequality (9), we deduce

$$\|(x, y)\omega\|_{L_t^\infty(L^p)} \lesssim \|(x, y)\omega^0\|_{L^p} + t\|\omega\|_{L_t^\infty(L^p)}\|u\|_{L_t^\infty(L^\infty)} \quad \forall p \in [1, \infty].$$

For concluded the proof stays to controlled  $\|(x, y)u\|_{L^2}$ , we have

$$\partial_t(xu) + (u \cdot \nabla)(xu) + \nabla(xp) = \begin{pmatrix} p + (u_1)^2 \\ u_1 u_2 \\ u_1 u_3 \end{pmatrix},$$

<sup>1</sup> We recall the classical fact  $\|\dot{\Delta}_q u\|_{L^p} \approx 2^{-q}\|\dot{\Delta}_q \omega\|_{L^p}$  uniformly in  $q$ , for every  $p \in [1, +\infty]$ .

from which, we deduce

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|xu\|_{L^2}^2 &= \int_{\mathbb{R}^2 \times ]-\pi, \pi[} xu_1 (2p + (u_1)^2 + (u_2)^2 + (u_3)^2) dx dy dz \\ &\lesssim \|xu\|_{L^2} (\|p\|_{L^2} + \|u\|_{L^\infty} \|u\|_{L^2}). \end{aligned}$$

As

$$-\Delta p = \operatorname{div} \operatorname{div}(u \otimes u),$$

Parseval's equality and the following inequality

$$\frac{2^{2j} + n^2 + |n|2^j}{n^2 + 2^{2j}} \lesssim 1$$

we follow the same approach in the proof of Lemma 3.1, we obtain

$$\|p\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} \lesssim \|u \otimes u\|_{L^2} \lesssim \|u\|_{L^\infty} \|u\|_{L^2}.$$

Then

$$(10) \quad \|xu\|_{L^2} \lesssim \|xu^0\|_{L^2} + t \|u\|_{L_t^\infty(L^2)} \|u\|_{L_t^\infty(L^\infty)} \lesssim \|xu^0\|_{L^2} + t \|u^0\|_{L^2} \|u\|_{L_t^\infty(L^\infty)} \leq C_0 e^{C_0 t}.$$

As a consequence, we obtain

$$\begin{aligned} \|xu\|_{L^\infty(\mathbb{R}^2 \times ]-\pi, \pi[)} &\leq \|\dot{S}_0(xu)\|_{L^\infty} + \sum_{q \geq 0} \|\dot{\Delta}_q(xu)\|_{L^\infty} \\ &\lesssim \|u\|_{L^2} + \|xu\|_{L^2} + \|\omega\|_{L^2} + \|x\omega\|_{L^2} \\ &\lesssim \|u^0\|_{L^2} + \|xu^0\|_{L^2} + (\|u^0\|_{L^2} + \|\omega^0\|_{L^\infty \cap L^2} + \|x\omega^0\|_{L^2}) e^{Ct \|\omega_z^0\|_{L^\infty \cap L^2}} \\ &\leq C(\|u^0\|_{L^2} + \|xu^0\|_{L^2} + \|\omega^0\|_{L^\infty \cap L^2} + \|x\omega^0\|_{L^2}) e^{Ct \|\omega_z^0\|_{L^\infty \cap L^2}}. \end{aligned}$$

And a similar argument gives the same estimate for  $\|yu\|_{L^\infty}$ . Hence the proposition.  $\square$

The evolution of the quantity  $\|\nabla u\|_{L_t^1(L^\infty)}$  is related to the following result:

**Proposition 4.2.** *There exists a decomposition  $(\tilde{\omega}_{q,n})_{(q,n) \in \mathbb{Z}^2}$  of the vorticity  $\omega$  such that*

i) *For every  $t \in \mathbb{R}_+$ , we have*

$$\omega = \sum_{(q,n) \in \mathbb{Z}^2} \tilde{\omega}_{q,n} e^{inz}$$

and

$$\operatorname{div} \tilde{\omega}_{q,n}(t, x) = 0.$$

ii) *For every  $(q, n) \in \mathbb{Z}^2$ , we have*

$$\|\tilde{\omega}_{q,n}(t)\|_{L^\infty} \lesssim (\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{C_0 t}$$

where  $C_0$  is a constant depending on  $u^0$  and  $c_q \in \ell^1(\mathbb{Z})$  (see Proposition 4.7).

iii) *For every  $(j, q, n) \in \mathbb{Z}^3$ , we have*

$$\|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C_0 2^{-|j-q|} e^{C_0 U(t)} (\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}),$$

with  $U(t) := \|\tilde{u}\|_{L_t^1(B_{\infty,1}^1)} + \|u\|_{L_t^1(\dot{B}_{2,1}^1)}$ .

*Proof.* We will localize in vertical frequency the initial data and denote by  $\tilde{\omega}_n$  the unique global vector-valued solution of the problem

$$\begin{cases} \partial_t \tilde{\omega}_{1,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,n} = -\tilde{\omega}_{z,n} \tilde{u}_2, \\ \partial_t \tilde{\omega}_{2,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,n} = \tilde{\omega}_{z,n} \tilde{u}_1, \\ \partial_t \tilde{\omega}_{z,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,n} = 0, \\ \tilde{\omega}_n|_{t=0} = \tilde{\omega}_n(0) \end{cases}$$

where

$$\tilde{\omega}_n(0) = \begin{pmatrix} -y\omega_{n,z}^0 \\ x\omega_{n,z}^0 \\ \omega_{n,z}^0 \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_1 + yu_3 \\ u_2 - xu_3 \end{pmatrix}, \quad \nabla_h = (\partial_x, \partial_y)$$

and

$$\omega_{n,z}^0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \omega_z^0 e^{-inz} dz = \partial_x u_{n,2}^0 - \partial_y u_{n,1}^0.$$

By Proposition 3.4, we deduce that  $\tilde{\omega}_n$  is the Fourier coefficients of  $\omega$ , i.e,  $\tilde{\omega}_n = \omega_n$ . Thus

$$\omega = \sum_{n \in \mathbb{Z}} \omega_n e^{inz} \quad \text{and} \quad \|\omega\|_{\dot{\mathcal{B}}_{p,r}^s} = \sum_{n \in \mathbb{Z}} \|\omega_n\|_{\dot{B}_{p,r}^s}.$$

We will use for this purpose a new approach similar to [13], which consists to linearize properly the Fourier of transport equation. For that, we will localize in frequency the initial data and denote by  $\tilde{\omega}_q$  the unique global vector-valued solution of the problem

$$\begin{cases} \partial_t \tilde{\omega}_{1,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,q,n} = -\tilde{\omega}_{z,q,n} \tilde{u}_2, \\ \partial_t \tilde{\omega}_{2,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,q,n} = \tilde{\omega}_{z,q,n} \tilde{u}_1, \\ \partial_t \tilde{\omega}_{z,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,q,n} = 0, \\ \tilde{\omega}_{q,n}|_{t=0} = \tilde{\omega}_{q,n}(0) \end{cases}$$

where

$$\tilde{\omega}_{q,n}(0) = \begin{pmatrix} -y\dot{\Delta}_q \omega_{n,z}^0 \\ x\dot{\Delta}_q \omega_{n,z}^0 \\ \dot{\Delta}_q \omega_{n,z}^0 \end{pmatrix}, \quad \tilde{u} = \begin{pmatrix} u_1 + yu_3 \\ u_2 - xu_3 \end{pmatrix}, \quad \nabla_h = (\partial_x, \partial_y)$$

and

$$\dot{\Delta}_q \omega_{n,z}^0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \dot{\Delta}_q \omega_z^0 e^{-inz} dz = \partial_x \dot{\Delta}_q u_{n,2}^0 - \partial_y \dot{\Delta}_q u_{n,1}^0.$$

In addition by linearity and uniqueness

$$\omega = \sum_{(q,n) \in \mathbb{Z}^2} \tilde{\omega}_{q,n} e^{inz}.$$

Since  $\operatorname{div} \tilde{\omega}_{q,n}(0) = -y\partial_x(\dot{\Delta}_q \omega_{z,n}^0) + x\partial_y(\dot{\Delta}_q \omega_{z,n}^0) = \partial_\theta(\dot{\Delta}_q \omega_{z,n}^0) = 0$  and  $\tilde{\omega}_q(0) = r\dot{\Delta}_q \omega_z^0 e_\theta + \dot{\Delta}_q \omega_z^0 e_z$ , then applying Proposition 3.3 gives  $\tilde{\omega}_{q,n} = r\tilde{\omega}_{q,z,n} e_\theta + \tilde{\omega}_{q,z,n} e_z$  and

$$(11) \quad \begin{cases} \partial_t \tilde{\omega}_{q,n} + (u \cdot \nabla) \tilde{\omega}_{q,n} = \tilde{\omega}_{q,n,z} (u_r e_\theta - u_\theta e_r) \\ \tilde{\omega}_{q,n}|_{t=0} = \tilde{\omega}_{q,n}(0). \end{cases}$$

Applying the maximum principle and using Propositions 4.1, 4.7, we obtain

$$(12) \quad \begin{aligned} \|\tilde{\omega}_{q,n}(t)\|_{L^\infty} &\lesssim \|\tilde{\omega}_{q,n}(0)\|_{L^\infty} + t \|\tilde{\omega}_{q,n,z}\|_{L_t^\infty(L^\infty)} \|u\|_{L_t^\infty(L^\infty)} \\ &\lesssim (\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{C_0 t} \\ &\lesssim 2^q (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}. \end{aligned}$$

This complete the proof of i)-ii) of the proposition.

Let us now move to the proof of iii) which is the main property of the above decomposition. Remark first that the desired estimate is equivalent to

$$(13) \quad \|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C 2^{j-q} e^{CU(t)} (\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0})$$

and

$$(14) \quad \|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C 2^{q-j} e^{CU(t)} (\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}),$$

with  $c_q \in \ell^1(\mathbb{Z})$ . From Corollary 3.1, it is plain that the  $\tilde{\omega}_{q,n}$  is solution of

$$(15) \quad \begin{cases} \partial_t \tilde{\omega}_{1,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,q,n} = \tilde{\omega}_{2,q,n} u_3 - \tilde{\omega}_{z,q,n} u_2, \\ \partial_t \tilde{\omega}_{2,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,q,n} = \tilde{\omega}_{z,q,n} u_1 - \tilde{\omega}_{1,q,n} u_3, \\ \partial_t \tilde{\omega}_{z,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,q,n} = 0, \end{cases}$$

with

$$\tilde{u} = (u_1 + y u_3, u_2 - x u_3) \quad \text{and} \quad \partial_x(u_1 + y u_3) + \partial_y(u_2 - x u_3) = 0.$$

*Step 1: Proof of (13).* Applying Corollary 4.2 to (15)

$$(16) \quad e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,\infty}^{-1}} \lesssim \|\tilde{\omega}_{q,n}(0)\|_{\dot{B}_{\infty,\infty}^{-1}} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{i,q,n} u_j(\tau)\|_{\dot{B}_{\infty,\infty}^{-1}} d\tau.$$

To estimate the integral term we write in view of Bony's decomposition

$$\begin{aligned} \|\tilde{\omega}_{i,q,n} u_j\|_{\dot{B}_{\infty,\infty}^{-1}} &\leq \|T_{\tilde{\omega}_{i,q,n}} u_j\|_{\dot{B}_{\infty,\infty}^{-1}} + \|T_{u_j} \tilde{\omega}_{i,q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} + \|R(\tilde{\omega}_{i,q,n}, u_j)\|_{\dot{B}_{\infty,\infty}^{-1}} \\ &\lesssim \|u\|_{L^\infty} \|\tilde{\omega}_{z,q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} + \|R(\tilde{\omega}_{i,q,n}, u_j)\|_{\dot{B}_{\infty,\infty}^{-1}}. \end{aligned}$$

The remainder term can be treated as follows

$$\begin{aligned} \|R(\tilde{\omega}_{i,q,n}, u_j)\|_{\dot{B}_{\infty,\infty}^{-1}} &\lesssim \sup_k \sum_{\ell \geq k-3} \|\dot{\Delta}_\ell \tilde{\omega}_{i,q,n}\|_{L^\infty} \|\tilde{\Delta}_\ell u_j\|_{L^2} \\ &\lesssim \|\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} \|u\|_{\dot{B}_{2,1}^1}. \end{aligned}$$

It follows that

$$\|\tilde{\omega}_{i,q,n} u_j\|_{\dot{B}_{\infty,\infty}^{-1}} \lesssim \|u\|_{\dot{B}_{2,1}^1} \|\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,\infty}^{-1}}.$$

Inserting this estimate into (16) we get

$$e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,\infty}^{-1}} \lesssim \|\tilde{\omega}_{q,n}(0)\|_{\dot{B}_{\infty,\infty}^{-1}} + \int_0^t \|u(\tau)\|_{\dot{B}_{2,\infty}^1} e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{q,n}(\tau)\|_{\dot{B}_{\infty,\infty}^{-1}} d\tau.$$

Hence we obtain by Gronwall's inequality and unsung Proposition 4.7

$$\begin{aligned} \|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,1}^{-1}} &\leq C(\|\dot{\Delta}_q \omega_n^0\|_{\dot{B}_{\infty,1}^{-1}} + \|\tilde{h}_q^1 * \omega_{z,n}^0\|_{B_{\infty,\infty}^{-1}} + \|\tilde{h}_q^2 * \omega_{z,n}^0\|_{B_{\infty,\infty}^{-1}}) \\ (17) \quad &\times e^{C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)} + C\|u\|_{L_t^1(\dot{B}_{2,1}^1)}} \\ &\leq C2^{-q}(\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{CU(t)}. \end{aligned}$$

This gives by definition

$$\|\dot{\Delta}_j \tilde{\omega}_q(t)\|_{L^\infty} \leq C2^{j-q}(\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{CU(t)}.$$

*Step 2: Proof of (14).* The solution  $\tilde{\omega}_q$  has three components in the cartesian basis  $\tilde{\omega}_{q,n} = (\tilde{\omega}_{1,q,n}, \tilde{\omega}_{2,q,n}, \tilde{\omega}_{z,q,n})$ . It's clear that  $\tilde{\omega}_{1,q,n}$  is solution of

$$\begin{cases} \partial_t \tilde{\omega}_{1,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,q,n} = -\tilde{\omega}_{z,q,n} \tilde{u}_2 \\ \tilde{\omega}_{1,q,n}|_{t=0} = \tilde{\omega}_{1,q,n}(0). \end{cases}$$

Then, we obtain from Corollary 4.2

$$e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{1,q,n}(t)\|_{\dot{B}_{\infty,1}^1} \lesssim \|\tilde{\omega}_{1,q,n}(0)\|_{\dot{B}_{\infty,1}^1} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n} \tilde{u}_2\|_{\dot{B}_{\infty,1}^1} d\tau.$$

From Bony's decomposition, we get

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{1,q,n}(t)\|_{\dot{B}_{\infty,1}^1} &\lesssim \|\tilde{\omega}_{1,q,n}(0)\|_{\dot{B}_{\infty,1}^1} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n}\|_{\dot{B}_{\infty,1}^1} \|\tilde{u}\|_{L^\infty} d\tau \\ &\quad + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{u}\|_{\dot{B}_{\infty,1}^1} \|\tilde{\omega}_{z,q,n}\|_{L^\infty} d\tau. \end{aligned}$$



The analysis will be exactly the same for  $\tilde{\omega}_{2,q,n}$ , because it satisfies the following equation

$$\begin{cases} \partial_t \tilde{\omega}_{2,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,q,n} = \tilde{\omega}_{2,q,n} \tilde{u}_1 \\ \tilde{\omega}_{2,q,n}|_{t=0} = \tilde{\omega}_{2,q,n}(0). \end{cases}$$

Since  $\tilde{\omega}_{z,q,n}$  satisfies

$$\begin{cases} \partial_t \tilde{\omega}_{z,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,q,n} = 0 \\ \tilde{\omega}_{z,q,n}|_{t=0} = \dot{\Delta}_q \omega_{z,n}^0, \end{cases}$$

then

$$e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n}(t)\|_{\dot{B}_{\infty,1}^1} \lesssim \|\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1}$$

and

$$\|\tilde{\omega}_{z,q,n}(t)\|_{L^\infty} \leq \|\dot{\Delta}_q \omega_{z,n}^0\|_{L^\infty} \lesssim 2^q \|\dot{\Delta}_q \omega_{z,n}^0\|_{L^2}.$$

Finally we obtain

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,1}^1} &\lesssim \|\dot{\Delta}_q \omega_n^0\|_{\dot{B}_{\infty,1}^1} + \|\tilde{h}_q^1 * \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1} + \|\tilde{h}_q^2 * \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1} \\ &\quad + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n}\|_{\dot{B}_{\infty,1}^1} \|\tilde{u}\|_{L^\infty} d\tau \\ &\quad + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{u}\|_{\dot{B}_{\infty,1}^1} \|\tilde{\omega}_{z,q,n}\|_{L^\infty} d\tau. \end{aligned}$$

So according to Gronwall's inequality and using Propositions 4.1 and 4.7 (see Appendix), we obtain

$$(18) \quad \|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,1}^1} \leq C2^q (\|\dot{\Delta}_q \omega^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{CU(t)}.$$

This can be written

$$\|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C2^{q-j} (\|\dot{\Delta}_q \omega^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{CU(t)}.$$

Hence, the desired result.  $\square$

So that  $x\tilde{\omega}_{q,n}$  and  $y\tilde{\omega}_{q,n}$  satisfies

$$\partial_t(x\tilde{\omega}_{q,n}) + (\tilde{u} \cdot \nabla_h)(x\tilde{\omega}_{q,n}) = \tilde{u}_1 \tilde{\omega}_{q,n} + \begin{pmatrix} -\tilde{u}_2 \tilde{\omega}_{2,q,n} \\ \tilde{u}_1 \tilde{\omega}_{2,q,n} \\ 0 \end{pmatrix}$$

and

$$\partial_t(y\tilde{\omega}_{q,n}) + (\tilde{u} \cdot \nabla_h)(y\tilde{\omega}_{q,n}) = \tilde{u}_2 \tilde{\omega}_{q,n} + \begin{pmatrix} \tilde{u}_2 \tilde{\omega}_{1,q,n} \\ -\tilde{u}_1 \tilde{\omega}_{1,q,n} \\ 0 \end{pmatrix}.$$

We follow the same proof of the previous proposition and using Corollary 4.3, we obtain the following proposition.

**Proposition 4.3.** *There exists  $C_0$  is a constant depending on  $u^0$  and  $c_q \in \ell^1(\mathbb{Z})$ , such that.*

i) *For every  $(q, n) \in \mathbb{Z}^2$ , we have*

$$\|(x, y)\tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C_0 (\|\dot{\Delta}_q \{(1, x, y)\omega_n^0\}\|_{L^\infty} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}.$$

ii) *For every  $(j, q, n) \in \mathbb{Z}^3$ , we have*

$$\begin{aligned} \|\dot{\Delta}_j \{(x, y)\tilde{\omega}_{q,n}\}(t)\|_{L^\infty} &\leq C_0 2^{-|j-q|} e^{C_0 U(t)} \exp(C_0 e^{C_0 t}) \\ &\quad \times \left( \|\dot{\Delta}_q \{(1, x, y)\omega_n^0\}\|_{L^\infty} + \|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} \right). \end{aligned}$$

with  $U(t) := \|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)} + \|u\|_{L_t^1(\dot{B}_{2,1}^1)}.$

*Proof.* i) According to maximum principle, Propositions 4.1, 4.2, 4.7 and Corollary 4.3, we have

$$\begin{aligned} \|(x, y)\tilde{\omega}_{q,n}(t)\|_{L^\infty} &\lesssim \|(x, y)\dot{\Delta}_q \omega_{z,n}^0\|_{L^\infty} + \|xy\dot{\Delta}_q \omega_{z,n}^0\|_{L^\infty} + \|x^2\dot{\Delta}_q \omega_{z,n}^0\|_{L^\infty} + \|y^2\dot{\Delta}_q \omega_{z,n}^0\|_{L^\infty} \\ &\quad + \int_0^t \|\tilde{u}\|_{L^\infty} \|\tilde{\omega}_{q,n}\|_{L^\infty} d\tau \\ &\lesssim (\|\dot{\Delta}_q \{(1, x, y)\omega_n^0\}\|_{L^\infty} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}. \end{aligned}$$

ii) Corollary 4.2, implies that

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|(x, y)\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,1}^1} &\lesssim \|(x, y)\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1} + \|xy\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1} + \|x^2\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1} \\ &\quad + \|y^2\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,1}^1} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{u}\|_{L^\infty} \|\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,1}^1} d\tau \\ &\quad + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{q,n}\|_{L^\infty} \|\tilde{u}\|_{\dot{B}_{\infty,1}^1} d\tau, \end{aligned}$$

this along with Propositions 4.1, 4.2, 4.7, Corollary 4.3 and inequalities (12), (18), ensures that

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|(x, y)\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,1}^1} &\lesssim 2^q (\|\dot{\Delta}_q \{(1, x, y)\omega_n^0\}\|_{L^\infty} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) \\ &\quad + 2^q (\|\dot{\Delta}_q \omega_n^0\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0}) e^{C_0 t + C\|u\|_{L_t^1(\dot{B}_{2,1}^1)}} \\ &\quad + 2^q (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_n^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}. \end{aligned}$$

As a consequence, we obtain

$$\begin{aligned} \|\dot{\Delta}_j \{(x, y)\tilde{\omega}_{q,n}\}(t)\|_{L^\infty} &\leq C_0 2^{q-j} e^{C_0(t+U(t))} \\ &\quad \times \left( \|\dot{\Delta}_q \{(1, x, y)\omega_n^0\}\|_{L^\infty} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} + \|\dot{\Delta}_q \omega_n^0\|_{L^2} \right). \end{aligned}$$

To prove the estimate

$$\begin{aligned} \|\dot{\Delta}_j \{(x, y)\tilde{\omega}_{q,n}\}(t)\|_{L^\infty} &\leq C_0 2^{j-q} e^{C_0(t+U(t))} \\ &\quad \times \left( \|\dot{\Delta}_q \{(1, x, y)\omega_n^0\}\|_{L^\infty} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} + \|\dot{\Delta}_q \omega_n^0\|_{L^2} \right), \end{aligned}$$

we use the fact that

$$\partial_t(x\tilde{\omega}_{q,n}) + (\tilde{u} \cdot \nabla_h)(x\tilde{\omega}_{q,n}) = u_1\tilde{\omega}_{q,n} + u_3(y\tilde{\omega}_{q,n}) + \begin{pmatrix} -u_2\tilde{\omega}_{2,q,n} + u_3(x\tilde{\omega}_{2,q,n}) \\ u_1\tilde{\omega}_{2,q,n} + u_3(y\tilde{\omega}_{2,q,n}) \\ 0 \end{pmatrix}$$

and

$$\partial_t(y\tilde{\omega}_{q,n}) + (\tilde{u} \cdot \nabla_h)(y\tilde{\omega}_{q,n}) = u_2\tilde{\omega}_{q,n} - u_3(x\tilde{\omega}_{q,n}) + \begin{pmatrix} u_2\tilde{\omega}_{1,q,n} - u_3(x\tilde{\omega}_{1,q,n}) \\ -u_1\tilde{\omega}_{1,q,n} - u_3(y\tilde{\omega}_{1,q,n}) \\ 0 \end{pmatrix}.$$

This along with Corollary 4.2 leads to

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|(x, y)\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,\infty}^{-1}} &\lesssim \|(x, y)\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} + \|xy\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} + \|x^2\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} \\ &\quad + \|y^2\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|u\|_{L^\infty} \|(1, x, y)\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} d\tau \\ &\quad + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|(1, x, y)\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} \|u\|_{\dot{B}_{2,1}^1} d\tau. \end{aligned}$$

Applying Gronwall's inequality, yields

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|(x, y)\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,\infty}^{-1}} &\lesssim \left\{ \|(x, y)\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} + \|xy\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} + \|x^2\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} \right. \\ &\quad + \|y^2\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{\infty,\infty}^{-1}} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|u\|_{L^\infty} \|\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} d\tau \\ &\quad \left. + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{q,n}\|_{\dot{B}_{\infty,\infty}^{-1}} \|u\|_{\dot{B}_{2,1}^1} d\tau \right\} e^{\|u\|_{L_t^1(L^\infty)} + \|u\|_{L_t^1(\dot{B}_{2,1}^1)}}. \end{aligned}$$

This along with Proposition 4.7, Corollary 4.3 and inequality (17), ensures that

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)}} \|(x, y)\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{\infty,\infty}^{-1}} &\leq C_0 2^{-q} e^{C_0 U(t)} \exp(C_0 e^{C_0 t}) \\ &\quad \times \left\{ \|\dot{\Delta}_q(1, x, y)\omega_n^0\|_{L^\infty} + c_q(\|\omega_n^0\|_{\dot{B}_{2,1}^0} + \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) \right\}. \end{aligned}$$

This completes the proof of Proposition 4.3.  $\square$

For conclude remnant to controlled  $\tilde{\omega}_{q,n}$  in  $\dot{B}_{2,1}^1$  and  $\dot{B}_{2,\infty}^{-1}$  (see Remark 3.1).

**Proposition 4.4.** *There exists  $C_0$  is a constant depending on  $u^0$  and  $c_q \in \ell^1(\mathbb{Z})$ , such that.*

i) *For every  $(q, n) \in \mathbb{Z}^2$ , we have*

$$\|\tilde{\omega}_{q,n}(t)\|_{L^2} + \|(1, x, y)\tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C_0 (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}.$$

ii) *For every  $(j, q, n) \in \mathbb{Z}^3$ , we have*

$$\|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^2} \leq C_0 2^{-|j-q|} e^{C_0 U(t)} \exp(C_0 e^{C_0 t}) (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}).$$

with  $U(t) := \|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)} + \|u\|_{L_t^1(\dot{B}_{2,1}^1)}$ .

*Proof.* It follows from (15) that

$$\begin{cases} \partial_t \tilde{\omega}_{1,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{1,q,n} = -\tilde{\omega}_{z,q,n} \tilde{u}_2 \\ \partial_t \tilde{\omega}_{2,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{2,q,n} = \tilde{\omega}_{z,q,n} \tilde{u}_1 \\ \partial_t \tilde{\omega}_{z,q,n} + (\tilde{u} \cdot \nabla_h) \tilde{\omega}_{z,q,n} = 0 \\ \tilde{\omega}_{1,q,n}|_{t=0} = \tilde{\omega}_{q,n}(0). \end{cases}$$

Taking  $L^2$  inner product of the above system with  $\tilde{\omega}_{q,n}$  gives

$$\frac{1}{2} \frac{d}{dt} \|\tilde{\omega}_{q,n}\|_{L^2}^2 \leq \|\tilde{\omega}_{q,n}\|_{L^2} \|\tilde{\omega}_{z,q,n}\|_{L^2} \|\tilde{u}\|_{L^\infty} \leq \|\dot{\Delta}_q \omega_{z,n}^0\|_{L^2} \|\tilde{\omega}_{q,n}\|_{L^2} \|\tilde{u}\|_{L^\infty}.$$

Applying Gronwall's inequality and using Propositions 4.1, 4.7 gives rise to

$$\|\tilde{\omega}_{q,n}(t)\|_{L^2} \leq C_0 (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}.$$

Thanks to Corollary 4.2, we obtain

$$\begin{aligned} e^{-C\|\tilde{u}\|_{L_t^1(\dot{B}_{2,1}^1)}} \|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{2,1}^1} &\lesssim \|\tilde{\omega}_{q,n}(0)\|_{\dot{B}_{2,1}^1} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n} \tilde{u}\|_{\dot{B}_{2,1}^1} d\tau \\ &\lesssim \|\tilde{\omega}_{q,n}(0)\|_{\dot{B}_{2,1}^1} + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n}\|_{\dot{B}_{2,1}^1} \|\tilde{u}\|_{L^\infty} d\tau \\ &\quad + \int_0^t e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n}\|_{L^2} \|\tilde{u}\|_{\dot{B}_{\infty,1}^1} d\tau. \end{aligned}$$

As  $\tilde{\omega}_{z,q,n}$  satisfies

$$e^{-C\|\tilde{u}\|_{L_\tau^1(\dot{B}_{\infty,1}^1)}} \|\tilde{\omega}_{z,q,n}\|_{\dot{B}_{2,1}^1} \leq \|\dot{\Delta}_q \omega_{z,n}^0\|_{\dot{B}_{2,1}^1} \lesssim 2^q \|\dot{\Delta}_q \omega_{z,n}^0\|_{L^2}$$

and

$$\|\tilde{\omega}_{z,q,n}\|_{L^2} \leq \|\dot{\Delta}_q \omega_{z,n}^0\|_{L^2} \lesssim 2^q c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}.$$

Then from Propositions 4.7 and 4.1 we find that for  $(q, n) \in \mathbb{Z}^2$

$$\|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{2,1}^1} \leq C_0 2^q (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0(t + \|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)})}.$$

By a similar proof of the previous inequality, we deduce

$$\|\tilde{\omega}_{q,n}(t)\|_{\dot{B}_{2,\infty}^{-1}} \leq C_0 2^{-q} (\|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C(\|\tilde{u}\|_{L_t^1(\dot{B}_{2,1}^1)} + \|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1))} \exp(C_0 e^{c_0 t}).$$

This completes the proof.  $\square$

So in conclusion, we obtain the following corollary.

**Corollary 4.1.** *For every  $(j, q, n) \in \mathbb{Z}^3$ , we have*

$$\|\tilde{\omega}_{q,n}(t)\|_{L^2} + \|\tilde{\omega}_{q,n}(t)\|_{L^\infty} \leq C_0 (\|\dot{\Delta}_q \{(1, x, y) \omega_n^0\}\|_{L^\infty} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}$$

and

$$\begin{aligned} \|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^2} + \|\dot{\Delta}_j \{(1, x, y) \tilde{\omega}_{q,n}\}(t)\|_{L^\infty} &\leq C_0 2^{-|j-q|} e^{C_0 U(t)} \exp(C_0 e^{C_0 t}) \\ &\times \left( \|\dot{\Delta}_q \{(1, x, y) \omega_n^0\}\|_{L^\infty} + \|\dot{\Delta}_q \omega_n^0\|_{L^2} + c_q \|\omega_n^0\|_{\dot{B}_{2,1}^0} + c_q \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} \right), \end{aligned}$$

with  $U(t) := \|\tilde{u}\|_{L_t^1(\dot{B}_{\infty,1}^1)} + \|u\|_{L_t^1(\dot{B}_{2,1}^1)}$  and  $c_q \in \ell^1(\mathbb{Z})$ .

**Proposition 4.5.** *The solution of (E) with initial data  $(1, x, y)u^0 \in L^2$ , such that  $\omega^0 \in \dot{\mathcal{B}}_{2,1}^0$ ,  $(1, x, y)\omega^0 \in \dot{\mathcal{B}}_{\infty,1}^0$  and  $\omega_z^0 \in \dot{\mathcal{B}}_{1,1}^0$  satisfies for every  $t \in \mathbb{R}_+$*

$$\|(1, x, y)u(t)\|_{L^2} \leq C_0 e^{C_0 t}$$

and

$$\|\omega_z(t)\|_{\dot{\mathcal{B}}_{1,1}^0} + \|\omega(t)\|_{\dot{\mathcal{B}}_{2,1}^0} + \|(1, x, y)\omega(t)\|_{\dot{\mathcal{B}}_{\infty,1}^0} \leq C_0 e^{\exp(e^{C_0 t})}$$

where  $C_0$  depends on the norms of  $u^0$ .

*Proof.* Inequality (10) implies the first estimate. Note that for any fixed integer  $N$ , one has

$$\begin{aligned} \|\omega_n(t)\|_{\dot{B}_{2,1}^0} + \|(1, x, y)\omega_n(t)\|_{\dot{B}_{\infty,1}^0} &\leq \sum_j \|\dot{\Delta}_j \sum_q (1, x, y) \tilde{\omega}_{q,n}(t)\|_{L^\infty} + \sum_j \|\dot{\Delta}_j \sum_q \tilde{\omega}_{q,n}(t)\|_{L^2} \\ &\leq \sum_{|j-q| \geq N} \|\dot{\Delta}_j \{(1, x, y) \tilde{\omega}_{q,n}\}(t)\|_{L^\infty} + \sum_{|j-q| \geq N} \|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^2} \\ &+ \sum_{|j-q| < N} \|\dot{\Delta}_j \{(1, x, y) \tilde{\omega}_{q,n}\}(t)\|_{L^\infty} + \sum_{|j-q| < N} \|\dot{\Delta}_j \tilde{\omega}_{q,n}(t)\|_{L^2} \\ (19) \quad &:= f_n + g_n \end{aligned}$$

Applying Corollary 4.1 gives

$$(20) \quad f_n \leq C_0 2^{-N} e^{C_0 U(t)} \exp(C_0 e^{C_0 t}) \left( \|(1, x, y)\omega_n^0\|_{\dot{B}_{\infty,1}^0} + \|\omega_n^0\|_{\dot{B}_{2,1}^0} + \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} \right)$$

and

$$(21) \quad g_n \leq C_0 N (\|(1, x, y)\omega_n^0\|_{\dot{B}_{\infty,1}^0} + \|\omega_n^0\|_{\dot{B}_{2,1}^0} + \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t}.$$

Combining this estimate with (19), (20) and (21), we obtain

$$\begin{aligned} \|\omega_n(t)\|_{\dot{B}_{2,1}^0} + \|(1, x, y)\omega_n(t)\|_{\dot{B}_{\infty,1}^0} &\lesssim N (\|(1, x, y)\omega_n^0\|_{\dot{B}_{\infty,1}^0} + \|\omega_n^0\|_{\dot{B}_{2,1}^0} + \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0}) e^{C_0 t} \\ &+ 2^{-N} e^{C_0 U(t)} \exp(C_0 e^{C_0 t}) \left( \|(1, x, y)\omega_n^0\|_{\dot{B}_{\infty,1}^0} + \|\omega_n^0\|_{\dot{B}_{2,1}^0} + \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} \right). \end{aligned}$$

Choosing the integer  $N$  so that

$$N \approx U(t),$$

leads to

$$\begin{aligned} \|\omega_n(t)\|_{\dot{B}_{2,1}^0} + \|(1, x, y)\omega_n(t)\|_{\dot{B}_{\infty,1}^0} &\lesssim U(t) \exp(C_0 e^{C_0 t}) \\ &\times \left( \|(1, x, y)\omega_n^0\|_{\dot{B}_{\infty,1}^0} + \|\omega_n^0\|_{\dot{B}_{2,1}^0} + \|\omega_{z,n}^0\|_{\dot{B}_{1,1}^0} \right). \end{aligned}$$

Lemmas 3.1, 3.2 and Remark 3.1 on the other hand, we have

$$\begin{aligned} \sum_{n \in \mathbb{Z}} \|\omega_n(t)\|_{\dot{B}_{2,1}^0} + \sum_{n \in \mathbb{Z}} \|(1, x, y)\omega_n(t)\|_{\dot{B}_{\infty,1}^0} &\lesssim (\|(1, x, y)\omega^0\|_{\dot{\mathcal{B}}_{\infty,1}^0} + \|\omega^0\|_{\dot{\mathcal{B}}_{2,1}^0} + \|\omega_z^0\|_{\dot{\mathcal{B}}_{1,1}^0}) \\ &\times \int_0^t (\|u\|_{L^2(\mathbb{R}^2 \times ]-\pi, \pi[)} + \|(1, x, y)\omega\|_{\dot{\mathcal{B}}_{\infty,1}^0} + \|\omega\|_{\dot{\mathcal{B}}_{2,1}^0}) d\tau \exp(C_0 e^{C_0 t}), \end{aligned}$$

hence we obtain by Gronwall's inequality

$$\|\omega(t)\|_{\dot{\mathcal{B}}_{2,1}^0} + \|(1, x, y)\omega(t)\|_{\dot{\mathcal{B}}_{\infty,1}^0} \leq C_0 e^{\exp(e^{C_0 t})}.$$

As  $\omega_z$  verifies

$$\partial_t \omega_z + (\tilde{u} \cdot \nabla_h) \omega_z = 0,$$

we obtain by Corollary 4.2

$$\|\omega_z\|_{\dot{\mathcal{B}}_{1,1}^0} \leq C_0 e^{\exp(e^{C_0 t})}.$$

This finishes the proof.  $\square$

**4.2. Existence and uniqueness.** The proof of existence of a solution is performed in a standard manner. We begin by solving an approximate problem, we are going to use Friedrich's method, which consists to approximation of system (E) by a truncation in the space of the frequencies. Let us define then the operator

$$J_{\ell,k} u = \sum_{|n| \leq k} e^{inz} 1_{r \leq \ell} \mathcal{F}^v u(n).$$

Let us consider the approximate equation

$$\partial_t u_{\ell,k} + J_{\ell,k} (J_{\ell,k} u_{\ell,k} \cdot \nabla) J_{\ell,k} u_{\ell,k} = J_{\ell,k} \mathcal{Q}(J_{\ell,k} u_{\ell,k}, u_{\ell,k})$$

with

$$\mathcal{Q}(u, v) = \sum_{i,j} \partial_i \partial_j (-\Delta)^{-1} (u^i v^j).$$

Later we prove that the solutions are uniformly bounded. The last step consists in studying the convergence to a solution of the initial equation. So we prove the local existence for regular data (for more details see [9, 10]). In critical spaces one can follow Park's approach in [16]. To prove that the solution associated to all helicoidal and enough smooth initial data  $u^0$ , is helicoidal, it suffices to use a method due to X. Saint Raymond [18]. In fact, it's clear that the first condition of Definition 1.1 is satisfied. Concerning the second condition, we have

$$\partial_t \{u(re_\theta + ke_z)\} + (u \cdot \nabla) \{u(re_\theta + ke_z)\} = \{(u \cdot \nabla)(re_\theta + ke_z)\} u - \nabla \Pi(re_\theta + ke_z)$$

i.e;

$$\partial_t \{u(re_\theta + ke_z)\} + (u \cdot \nabla) \{u(re_\theta + ke_z)\} = 0.$$

To prove the uniqueness simply to controlled the difference of two solutions in  $L^2(\mathbb{R}^2 \times ]-\pi, \pi[)$ .

## APPENDIX

To prove main theorem, we need some inequalities.

**Proposition 4.6.** *Let  $(r, p) \in [1, +\infty]^2$ ,  $f \in \dot{B}_{p,r}^s$ ,  $v$  be a smooth divergence free vector field and*

$$[v \cdot \nabla, \dot{\Delta}_q]f = (v \cdot \nabla)\dot{\Delta}_q f - \dot{\Delta}_q((v \cdot \nabla)f).$$

Then there hold

(i) If  $s = -1$

$$\sup_q 2^{-q} \|[v \cdot \nabla, \dot{\Delta}_q]f\|_{L^p} \lesssim \|\nabla v\|_{\dot{B}_{\infty,1}^0} \|f\|_{\dot{B}_{p,\infty}^{-1}}$$

(ii) If  $-1 < s < 1$

$$\left( \sum_q 2^{qsr} \|[v \cdot \nabla, \dot{\Delta}_q]f\|_{L^p}^r \right)^{\frac{1}{r}} \lesssim \|\nabla v\|_{L^\infty} \|f\|_{\dot{B}_{p,r}^s}.$$

(iii) If  $s = 1$

$$\sum_q 2^q \|[v \cdot \nabla, \dot{\Delta}_q]f\|_{L^p} \lesssim \|\nabla v\|_{\dot{B}_{\infty,1}^0} \|f\|_{\dot{B}_{p,1}^1}.$$

If  $f = \operatorname{rot} u$ , then the second point holds true for  $s \in [1, \infty[$ .

*Proof.* Thanks to Bony's decomposition, we write

$$[v \cdot \nabla, \dot{\Delta}_q]f = [T_{vj}, \dot{\Delta}_q]\partial_j f + T_{\partial_j \dot{\Delta}_q f} v^j - \dot{\Delta}_q T_{\partial_j f} v^j + R(v^j, \dot{\Delta}_q \partial_j f) - \dot{\Delta}_q R(v^j, \partial_j f).$$

For every  $s \in \mathbb{R}$ , by a classical inequality about commutators we have (see for example [7])

$$\begin{aligned} \left\{ \sum_q 2^{qsr} \|[T_{vj}, \dot{\Delta}_q]\partial_j f\|_{L^p}^r \right\}^{\frac{1}{r}} &\leq \left\{ \sum_q 2^{qsr} \|[T_{vj}, \dot{\Delta}_q]\partial_j f\|_{L^p}^r \right\}^{\frac{1}{r}} \\ &\lesssim \|\nabla v\|_{L^\infty} \|f\|_{\dot{B}_{p,r}^s}. \end{aligned}$$

For the paraproduct term, we have

$$T_{\partial_j \dot{\Delta}_q f} v^j = \sum_{q \leq q'} S_{q'-1} \dot{\Delta}_q \partial_j f \dot{\Delta}_{q'} v^j,$$

Applying Bernstein and Hölder inequalities leads to

$$\begin{aligned} \left\| \sum_{q \leq q'} S_{q'-1} \dot{\Delta}_q \partial_j f \dot{\Delta}_{q'} v^j \right\|_{L^p} &\lesssim \sum_{q \leq q'} \|\dot{\Delta}_{q'} v^j\|_{L^\infty} \|\dot{\Delta}_q \partial_j f\|_{L^p} \\ &\lesssim \sum_{q \leq q'} 2^{q-q'} \|\dot{\Delta}_{q'} \nabla v\|_{L^\infty} \|\dot{\Delta}_q f\|_{L^p}. \end{aligned}$$

Therefore

$$\left\{ \sum_q 2^{qsr} \|T_{\partial_j \dot{\Delta}_q f} v^j\|_{L^p}^r \right\}^{\frac{1}{r}} \lesssim \|\nabla v\|_{\dot{B}_{\infty,\infty}^0} \|f\|_{\dot{B}_{p,r}^s} \quad \forall s \in \mathbb{R}.$$

Concerning the term  $\dot{\Delta}_q T_{\partial_j f} v^j$ , we have

$$\dot{\Delta}_q T_{\partial_j f} v^j = \sum_{|q-q'| \leq 4} \Delta_q (S_{q'-1} \partial_j f \dot{\Delta}_{q'} v^j).$$

From the definition of  $\dot{S}_{q'-1}$  and applying Bernstein inequality, we obtain

$$2^{qs} \|\dot{\Delta}_q T_{\partial_j f} v^j\|_{L^p} \lesssim \sum_{|q-q'| \leq 4} \|\dot{\Delta}_{q'} \nabla v\|_{L^\infty} \sum_{k \leq q'-2} 2^{(k-q')(1-s)} 2^{ks} \|\Delta_k f\|_{L^p}.$$

Thus

$$\left( \sum_q 2^{qsr} \|\dot{\Delta}_q T_{\partial_j f} v^j\|_{L^p}^r \right)^{\frac{1}{r}} \lesssim \|\nabla v\|_{\dot{B}_{\infty,\infty}^0} \|f\|_{\dot{B}_{p,r}^s} \quad \forall s < 1$$

and for  $s = 1$

$$\sum_q 2^q \|\dot{\Delta}_q T_{\partial_j f} v^j\|_{L^p} \lesssim \|\nabla v\|_{\dot{B}_{\infty,1}^0} \|f\|_{\dot{B}_{p,1}^1}.$$

Finally divergence free of  $v$ , implies

$$R(v^j, \dot{\Delta}_q \partial_j f) - \dot{\Delta}_q R(v^j, \partial_j f) = \sum_{q' \geq q-3} \dot{\Delta}_{q'} v^j \tilde{\Delta}_{q'} \dot{\Delta}_q \partial_j f - \dot{\Delta}_q \partial_j (\dot{\Delta}_{q'} v^j \tilde{\Delta}_{q'} f) := I_q.$$

From Hölder and Bernstein inequalities, we deduce

$$\begin{aligned} \left( \sum_q 2^{qsr} \|I_q\|_{L^p}^r \right)^{\frac{1}{r}} &\lesssim \|\nabla v\|_{\dot{B}_{\infty,\infty}^0} \|f\|_{\dot{B}_{p,r}^s} \quad \text{if } s > -1, \\ \sup_q 2^{-q} \|I_q\|_{L^p} &\lesssim \|\nabla v\|_{\dot{B}_{\infty,1}^0} \|f\|_{\dot{B}_{p,1}^{-1}} \quad \text{if } s = -1. \end{aligned}$$

The proof is now achieved.  $\square$

An immediate corollary of the above lemma is.

**Corollary 4.2.** *Let  $s \in ]-1, 1[$ ,  $(p, r) \in [1, \infty]^2$  and  $u$  be a smooth divergence free vector field. Let  $f$  a smooth solution of the transport equation.*

$$\partial_t f + u \cdot \nabla f = g, \quad f|_{t=0} = f_0,$$

such that  $f_0 \in \dot{B}_{p,r}^s(\mathbb{R}^2)$  and  $g \in L_{loc}^1(\mathbb{R}_+; \dot{B}_{p,r}^s)$ . Then

$$(22) \quad \|f(t)\|_{\dot{B}_{p,r}^s} \leq C e^{CU(t)} (\|f_0\|_{\dot{B}_{p,r}^s} + \int_0^t e^{-CU(\tau)} \|g(\tau)\|_{\dot{B}_{p,r}^s} d\tau) \quad \forall t \in \mathbb{R}_+.$$

where  $U(t) = \int_0^t \|\nabla u(\tau)\|_{L^\infty} d\tau$  and  $C$  constant depends on  $s$ .

The above estimate holds also true in the limit cases :

$$s = -1, r = \infty, p \in [1, \infty] \text{ or } s = 1, r = 1, p \in [1, \infty],$$

provided that we change  $U(t)$  by  $U_1(t) := \|u\|_{L_t^1 \dot{B}_{\infty,1}^1}$ .

In addition if  $f = \text{rot } u$ , then the above estimate (22) holds true for all  $s \in [1, +\infty[$ .

**Proposition 4.7.** *Let  $f \in \mathcal{S}'(\mathbb{R}^2)$ , then*

i)

$$x_1 \dot{\Delta}_j f = \dot{\Delta}_j (y_1 f) + \tilde{h}_j^1 * f \quad \text{and} \quad x_2 \dot{\Delta}_j f = \dot{\Delta}_j (y_2 f) + \tilde{h}_j^2 * f$$

with

$$\tilde{h}_j^1(x) = 2^{-j} x_1 h(2^j x) \quad \text{and} \quad \tilde{h}_j^2(x) = 2^{-j} x_2 h(2^j x).$$

In addition

$$\dot{\Delta}_q (x_1 \dot{\Delta}_j \omega) = \dot{\Delta}_q (x_2 \dot{\Delta}_j \omega) = 0, \quad \text{for } |j - q| \geq 5.$$

ii) If  $f \in \dot{B}_{2,1}^0 \cap \dot{B}_{1,1}^0$ , then

$$\begin{aligned} \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j (y_1 f)\|_{L^\infty} + \|x_2 \dot{\Delta}_j f - \dot{\Delta}_j (y_2 f)\|_{L^\infty} &\lesssim c_j \begin{cases} \|f\|_{\dot{B}_{2,1}^0}, \\ 2^j \|f\|_{\dot{B}_{1,1}^0}, \end{cases} \\ \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j (y_1 f)\|_{\dot{B}_{\infty,1}^1} + \|x_2 \dot{\Delta}_j f - \dot{\Delta}_j (y_2 f)\|_{\dot{B}_{\infty,1}^1} &\lesssim c_j 2^j \|f\|_{\dot{B}_{2,1}^0}, \\ \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j (y_1 f)\|_{\dot{B}_{\infty,1}^{-1}} + \|x_2 \dot{\Delta}_j f - \dot{\Delta}_j (y_2 f)\|_{\dot{B}_{\infty,1}^{-1}} &\lesssim c_j 2^{-j} \|f\|_{\dot{B}_{2,1}^0}, \\ \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j (y_1 f)\|_{L^2} + \|x_2 \dot{\Delta}_j f - \dot{\Delta}_j (y_2 f)\|_{L^2} &\lesssim c_j \|f\|_{\dot{B}_{1,1}^0}, \end{aligned}$$

$$\begin{aligned} \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j(y_1 f)\|_{\dot{B}_{2,1}^1} + \|x_2 \dot{\Delta}_j f - \dot{\Delta}_j(y_2 f)\|_{\dot{B}_{2,1}^1} &\lesssim c_j 2^j \|f\|_{\dot{B}_{1,1}^0}, \\ \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j(y_1 f)\|_{\dot{B}_{2,1}^{-1}} + \|x_2 \dot{\Delta}_j f - \dot{\Delta}_j(y_2 f)\|_{\dot{B}_{2,1}^{-1}} &\lesssim c_j 2^{-j} \|f\|_{\dot{B}_{1,1}^0} \end{aligned}$$

with  $c_j \in \ell^1(\mathbb{Z})$ .

*Proof.* i) We write by definition

$$\begin{aligned} x_1 \dot{\Delta}_j f(x) - \dot{\Delta}_j(y_1 f)(x) &= 2^{2j} \int_{\mathbb{R}^2} h(2^j(x-y))(x_1 - y_1) f(y) dy \\ &= 2^{2j} \tilde{h}_j^1 \star f(x), \end{aligned}$$

with  $\tilde{h}_j^1(x) = 2^{-j} x_1 h(2^j x)$ . This complete the proof of i)

ii) Now we claim that for every  $f \in \mathcal{S}'$  we have

$$2^{2j} \tilde{h}(2^j \cdot) \star f = \sum_{|j-k| \leq 1} 2^{2j} \tilde{h}(2^j \cdot) \star \dot{\Delta}_k f.$$

Indeed, we have  $\widehat{\tilde{h}}(\xi) = i \partial_{\xi_1} \widehat{h}(\xi) = i \partial_{\xi_1} \varphi(\xi)$ . It follows that  $\text{supp } \widehat{\tilde{h}} \subset \text{supp } \varphi$ . So we get

$$2^{2j} \tilde{h}(2^j \cdot) \star \dot{\Delta}_k f = 0, \quad \text{for } |j-k| \geq 2.$$

This leads to

$$\begin{aligned} \sum_{j \in \mathbb{Z}} \|x_1 \dot{\Delta}_j f - \dot{\Delta}_j(y_1 f)\|_{L^\infty} &\lesssim \sum_{|j-k| \leq 1} 2^{k-j} 2^{-k} \|\dot{\Delta}_k f\|_{L^\infty} \\ &\lesssim \|f\|_{\dot{B}_{\infty,1}^{-1}} \\ &\lesssim \|f\|_{\dot{B}_{2,1}^0}. \end{aligned}$$

Similar for same inequalities. The proof is now achieved.  $\square$

We follow the same proof of the previous proposition and we use the fact that

$$x_i x_j - y_i y_j = (x_i - y_i)(x_j - y_j) + (x_i - y_i)y_j + (x_j - y_j)y_i,$$

we obtain the following corollary.

**Corollary 4.3.** *If  $(f, y_i f) \in \dot{B}_{1,1}^0 \times \dot{B}_{2,1}^0$ , then for  $1 \leq i, j \leq 2$ , we have*

$$\begin{aligned} \|x_i x_j \dot{\Delta}_j f - \dot{\Delta}_j(y_i y_j f)\|_{L^\infty} &\lesssim c_j (\|f\|_{\dot{B}_{1,1}^0} + \|y_1 f\|_{\dot{B}_{2,1}^0} + \|y_2 f\|_{\dot{B}_{2,1}^0}), \\ \|x_i x_j \dot{\Delta}_j f - \dot{\Delta}_j(y_i y_j f)\|_{\dot{B}_{\infty,1}^1} &\lesssim c_j 2^j (\|f\|_{\dot{B}_{1,1}^0} + \|y_1 f\|_{\dot{B}_{2,1}^0} + \|y_2 f\|_{\dot{B}_{2,1}^0}) \end{aligned}$$

and

$$\|x_i x_j \dot{\Delta}_j f - \dot{\Delta}_j(y_i y_j f)\|_{\dot{B}_{\infty,1}^{-1}} \lesssim c_j 2^{-j} (\|f\|_{\dot{B}_{1,1}^0} + \|y_1 f\|_{\dot{B}_{2,1}^0} + \|y_2 f\|_{\dot{B}_{2,1}^0})$$

with  $c_j \in \ell^1(\mathbb{Z})$ .

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